Engineering Beam Theory for the First Order Analysis with Finite Element Method

1998 Winter, Kikuchi

Slender structures whose length is much larger than the size of the cross section, are called beams. In such structures, deformation may be decomposed into

1. axial deformation
2. bending deformation in two directions
3. torsional deformation

In order to describe the engineering beam theory, we shall introduce the Cartesian coordinate system (x,y,z), where the z axis coincides with the beam axis define by the line formed by the centroid of the cross section, while the x and y axes are the principal axes of the cross section. The origin is set up at the centroid of the left edge cross section of the beam. Based on

a) the cross section does not deform, i.e., $\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$
b) Bernoulli-Euler assumption that no shearing strain is generated by pure bending
c) Saint-Venant torsion theory

the displacement of an arbitrary point (x,y,z) of the beam is approximated by

$$U(x,y,z) = u_s(z) - (y - y_s)\theta(z)$$
$$V(x,y,z) = v_s(z) + (x - x_s)\theta(z)$$
$$W(x,y,z) = w(z) - xu_s'(z) - yv_s'(z) + \omega_{ns}(x,y)\theta'(z)$$

where g' is the derivative of g in z,

$(u_s(z), v_s(s))$ transverse deflections of the shear center (x_s, y_s) in the x and y direction, respectively,

$w(z)$ average axial displacement

$\theta(z)$ angle of twist of the cross section at z

$\omega_{ns}(x,y)$ normalized Saint-Venant warping function when torque is applied about the shear center axis (that is the line passing through the shear center (x_s, y_s)), and is defined a property attached to the cross section such that

$$\omega_{ns}(x,y) = \omega_n(x,y) - y_s x + x_s y$$

$$\int_A \omega_{ns} dA = \int_A \omega_{ns} x dA = \int_A \omega_{ns} y dA = 0$$

$\omega_n(x,y)$ normalized warping function when torque is applied at the centroid axis (that is the beam axis, the line passing through the centroid of the cross section)

The beam theory based on the displacement approximation stated in above is called the engineering beam theory.

Using the assumed displacement field, strains are calculated as

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$

$$\varepsilon_z = w' - xu_s'' - yv_s'' + \omega_{ns}\theta''$$

$$\gamma_{zx} = \left(\frac{\partial \omega_{ns}}{\partial x} - (y - y_s) \right)\theta'$$

$$\gamma_{zy} = \left(\frac{\partial \omega_{ns}}{\partial y} + (x - x_s) \right)\theta'$$

Assuming Hook's low

$$\sigma_z = E\varepsilon_z \quad , \quad \tau_{zx} = G\gamma_{zx} \quad , \quad \tau_{zy} = G\gamma_{zy},$$

the total strain energy U_e stored in the beam is given by

$$U_e = \frac{1}{2}\int_V \left(\sigma_z\varepsilon_z + \tau_{zx}\gamma_{zx} + \tau_{zy}\gamma_{zy}\right)dV$$

$$= \frac{1}{2}\int_0^l \int_A \left(\sigma_z\varepsilon_z + \tau_{zx}\gamma_{zx} + \tau_{zy}\gamma_{zy}\right)dAdz$$

$$= \frac{1}{2}\int_0^l \left\{EAw'^2 + EI_{xx}u_s''^2 + EI_{yy}v_s''^2 + \left(EI_\omega^{(s)}\theta''^2 + GK\theta'^2\right)\right\}dz$$

where

$A = \int_A dA$ cross sectional area

$$I_{xx} = \int_A x^2 dA \quad \text{moment of inertia about the y axis}$$

$$I_{yy} = \int_A y^2 dA \quad \text{moment of inertia about the x axis}$$

$$I_\omega^{(s)} = \int_A \omega_{ns}{}^2 dA \qquad \text{warping moment}$$

$$K = \int_A \left\{ \left(\frac{\partial \omega_{ns}}{\partial x} - (y - y_s) \right)^2 + \left(\frac{\partial \omega_{ns}}{\partial y} + (x - x_s) \right)^2 \right\} dA \qquad \text{Saint-Venant}$$

torsion constant.

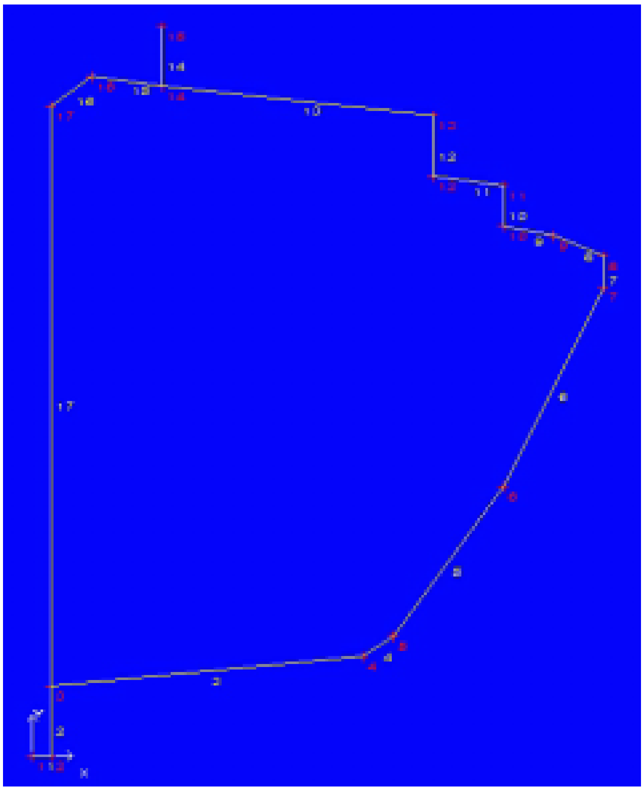
Here the x and y axes are the principal axes of the cross section such that

$$\int_A x dA = \int_A y dA = \int_A xy dA = 0 \, .$$

As an example, we shall consider the cross section shown in Fig. 1 that is a typical configuration of the side member of the car body frame structure. In this example, (x,y) is a coordinate ststem to define geometry of the cross section of the beam, and is not the principal axes of the cross section as in above. Section properties are computed by AISI CARS’96, GAS Program[1]. Geometry of the cross section is given in the output of CARS’96-GAS as follows :

---

[1] AISI/CARS’96 for Window : First Order Analysis in Automotive Steel Design, Auto/Steel Partnership, 2000 Town Center, 19th Floor, Southfield, MI 48075, (248) 351-2664. CARS’96 consists of four modules : Key (The Key to Automotive Steel Design), GAS (Geometric Analysis of Sections), MAP (Material Archive Program), and ASDM (Automotive Steel Design Manual).



GAS - CARS Geometric Analysis of Sections
Version 5.0

Date: Jan 20 1998   Time: 11:12:13
Units   : N, mm, MPa
Database   : C:\CARS96\USER\SIDEBAR
Section Name: sidemember
Description :

Cross Section Geometry:

Point No.	X	Y
1	0	0
2	2	0
3	2	7
4	33	10
5	36	12
6	47	27
7	57	47
8	57	50
9	52	52
10	47	53
11	47	57
12	40	58
13	40	64
14	13	67
15	13	73
16	6	68
17	2	65

Line No.	Start Pt.	End Pt.	Length	Thickness	Material Archive	No.	Line Type
1	1	2	2	1	asdm.	1	Segment
2	2	3	7	2	asdm.	1	Segment
3	3	4	31.1448	1	asdm.	1	Segment
4	4	5	3.60555	1	asdm.	1	Segment
5	5	6	18.6011	1	asdm.	1	Segment
6	6	7	22.3607	1	asdm.	1	Segment
7	7	8	3	1	asdm.	1	Segment
8	8	9	5.38516	1	asdm.	1	Segment
9	9	10	5.09902	1	asdm.	1	Segment
10	10	11	4	1	asdm.	1	Segment
11	11	12	7.07107	1	asdm.	1	Segment
12	12	13	6	1	asdm.	1	Segment
13	13	14	27.1662	1	asdm.	1	Segment
14	14	15	6	1	asdm.	1	Segment
15	14	16	7.07107	1	asdm.	1	Segment
16	16	17	5	1	asdm.	1	Segment
17	17	3	58	1	asdm.	1	Segment

Material Description:

Archive	Mat'l No.	E	Fy	Archive Location
asdm.	1	203000	234.422	c:\cars96\

*** Results ***

Nominal Properties:

```
Area  =       225.5
cx    =      22.561
cy    =        36.9
Ixx   =  1.1938E+05
Iyy   =       87317
Ixy   =       18950
Sx+   =      3306.9
Sy+   =      2499.2
Sx-   =       -3192
Sy-   =     -3870.2
Theta =     -24.884
Iuu   =  1.2817E+05
Ivv   =       78528
Su+   =      3956.3
Sv+   =      2856.5
Su-   =     -2951.8
Sv-   =     -2538.9
rx    =      23.009
ry    =      19.678
J     =  1.2743E+05
Cw    =           0
ex    =      24.561
ey    =      37.234
Cuu   =      2.6574
Cvv   =      2.1266
Jopen =      21.333
Jc    =  1.2741E+05
tomax =           2
tcmin =           1
Ao    =        2546
```

In this example, the principal axes are defined by the *u* and *v* axes, and then

$$I_{xx}\left(=I_{uu}\right)=1.2817E+05mm^4$$
$$I_{yy}\left(=I_{vv}\right)=7.8528E+04mm^4$$
$$K(=J)=1.2743E+05mm^4$$

and others.

In the above theory,

axial deformation (w)
bending about the y axis (u_s)
bending about the x axis (v_s)
torsional deformation (θ)

are independent each other. Thus we can consider these separately.

(1) Axial Bar (Truss) Element

In this case, we assume

$$U(x,y,z)=V(x,y,z)=0 \quad , \quad W(x,y,z)=w(z)$$

and the average axial displacement w varies linearly in the bar element in z :

$$w(z)=c_1+c_2z=\{1 \quad z\}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}=\boldsymbol{b}^T\boldsymbol{c}$$

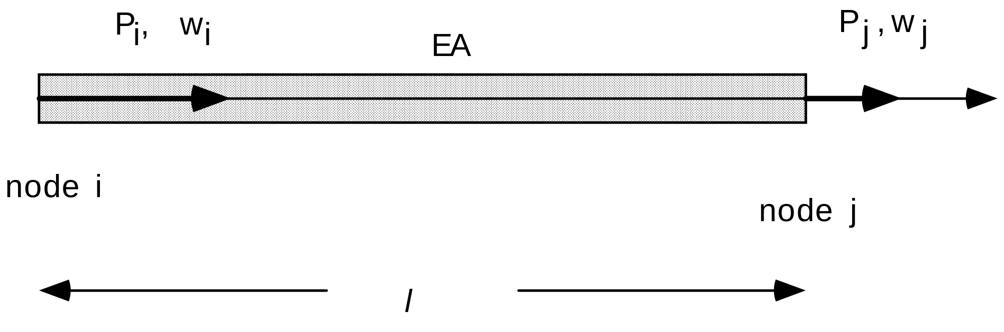


Figure 2 Axial Bar Element (Truss Element)

Noting that

$$\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}=\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}=\frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}$$

we have

$$w(z)=\{1 \quad z\}\frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}=\begin{Bmatrix} 1-\frac{z}{l} & \frac{z}{l} \end{Bmatrix}\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}=\boldsymbol{N}\boldsymbol{d}_w$$

and

$$\varepsilon_z=\frac{\partial w}{\partial z}=\frac{1}{l}\{-1 \quad 1\}\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}=\boldsymbol{B}\boldsymbol{d}_w$$

This yields the total strain energy

$$U_e=\frac{1}{2}\int_0^l EA\varepsilon_z{}^2 dz=\frac{1}{2}\int_0^l EA\left(\boldsymbol{B}\boldsymbol{d}_w\right)^T\boldsymbol{B}\boldsymbol{d}_w dz=\frac{1}{2}\boldsymbol{d}_w{}^T\boldsymbol{k}_w\boldsymbol{d}_w$$

where

$$\boldsymbol{k}_w=\int_0^l \boldsymbol{B}^T EA\boldsymbol{B}dz=\frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The corresponding generalized force vector is defined by the axial forces P_i and P_j :

$$\boldsymbol{f}_w=\begin{Bmatrix} P_i \\ P_j \end{Bmatrix}$$

and the discrete form of equilibrium becomes

$$\boldsymbol{k}_w\boldsymbol{d}_w=\boldsymbol{f}_w .$$

(2) Bending Element

For the bending about the y axis, we consider the displacement field

$$U(x,y,z)=u_s(z) \quad , \quad V(x,y,z)=0 \quad , \quad W(x,y,z)=-xu_s'(z)$$

and we shall assume the deflection us(z) of the shear center on the z cross section in the x direction by a third degree polynomial in z :

$$u_s(z)=\{1 \quad z \quad z^2 \quad z^3\}\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}.$$

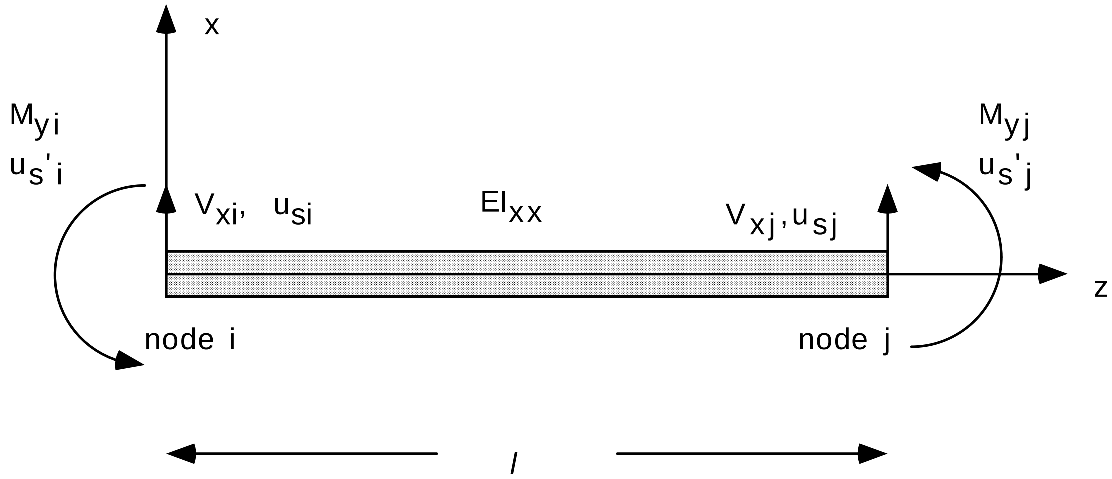


Figure 3 Beam Element (Moment Applied about the y axis)

If the generalized displacement vector of the bending element is given by

$$\boldsymbol{d}_u = \begin{Bmatrix} u_{si} \\ u_s'_i \\ u_{sj} \\ u_s'_j \end{Bmatrix}$$

where they are the deflection and slop at the beam end points *i* and *j*, respectively, we have the following relation :

$$\boldsymbol{d}_u = \begin{Bmatrix} u_{si} \\ u_s'_i \\ u_{sj} \\ u_s'_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \boldsymbol{d}_u$$

and

$$u_s(z) = \begin{Bmatrix} 1 & z & z^2 & z^3 \end{Bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \boldsymbol{d}_u = \boldsymbol{N}\boldsymbol{d}_u \,.$$

Using this the normal strain is given by

$$\varepsilon_z = \frac{\partial W}{\partial z} = -x\frac{d^2u_s}{dz^2} = -x\{0 \quad 0 \quad 2 \quad 6z\}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \boldsymbol{d}_u = -x\boldsymbol{B}\boldsymbol{d}_u$$

where

$$\boldsymbol{B} = \left\{ -\frac{6}{l^2} + \frac{12z}{l^3} \quad -\frac{4}{l} + \frac{6z}{l^2} \quad \frac{6}{l^2} - \frac{12z}{l^3} \quad -\frac{2}{l} + \frac{6z}{l^2} \right\}$$

and the strain energy stored in the beam element becomes

$$U_e = \frac{1}{2}\int_0^l EI_{xx}(u_s'')^2 dz = \frac{1}{2}\boldsymbol{d}_u{}^T \int_0^l \boldsymbol{B}^T EI_{xx}\boldsymbol{B}dz\boldsymbol{d}_u = \frac{1}{2}\boldsymbol{d}_u{}^T \boldsymbol{k}_u \boldsymbol{d}_u$$

where

$$\boldsymbol{k}_u = \int_0^l \boldsymbol{B}^T EI_{xx}\boldsymbol{B}dz = \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & & & sym \\ 6l & 4l^2 & & \\ -12 & -6l & 12 & \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}.$$

The corresponding generalized force vector is defined by the transverse forces V_{xi}, V_{xj} and bending moments M_{yi}, M_{yj} about the y axis at the two end points I and j of the beam element, respectively :

$$\boldsymbol{f}_u = \begin{Bmatrix} V_{xi} \\ M_{yi} \\ V_{xj} \\ M_{yj} \end{Bmatrix}.$$

Then the discrete form of equilibrium becomes

$$\boldsymbol{k}_u\boldsymbol{d}_u = \boldsymbol{f}_u.$$

Next we shall consider the bending v_s about the x axis :

$$U(x,y,z) = \quad , \quad V(x,y,z) = v_s(z) \quad , \quad W(x,y,z) = -yv_s'(z)$$

where $v_s(z)$ is the deflection of the shear center of the z cross section in the y direction.

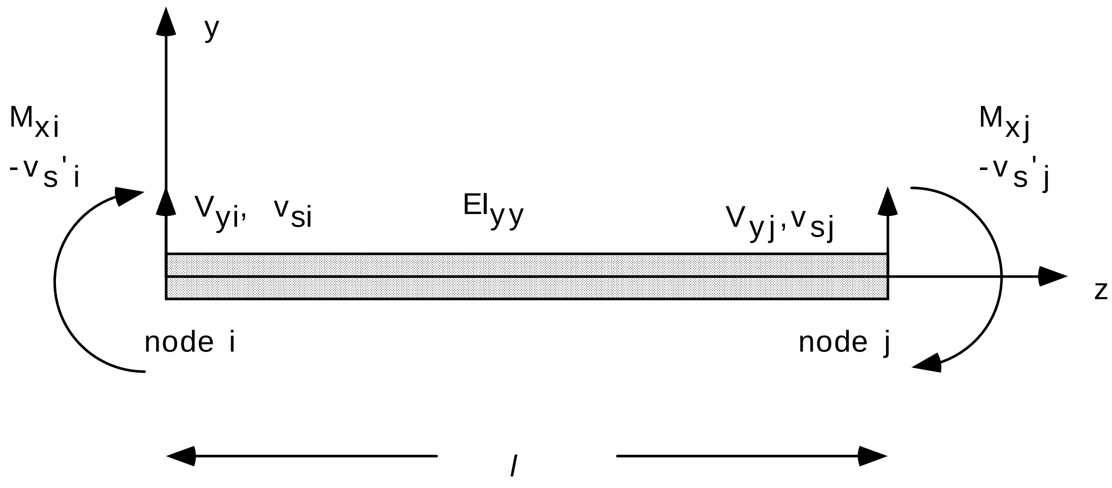


Figure 4 Beam Element (Moment Applied About the x Axis)

Noting that the positive bending moment about the x axis coincides with the negative slope of the beam axis, the generalized displacement vector $\boldsymbol{d}_v$ must be defined by

$$\boldsymbol{d}_v = \begin{Bmatrix} v_{si} \\ -v_s{'}_i \\ v_{sj} \\ -v_s{'}_j \end{Bmatrix}$$

which is associated with the generalized force vector $\boldsymbol{f}_v$:

$$\boldsymbol{f}_v = \begin{Bmatrix} V_{yi} \\ M_{xi} \\ V_{yj} \\ M_{xj} \end{Bmatrix}.$$

Thus, the element stiffness matrix $\boldsymbol{k}_v$ can be written by the similar form of $\boldsymbol{k}_u$ after exchanging the sign of the second and fourth columns and rows with the moment of inertia I_{yy} :

$$\boldsymbol{k}_v = \frac{EI_{yy}}{l^3}\begin{bmatrix} 12 & & & sym \\ -6l & 4l^2 & & \\ -12 & 6l & 12 & \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix}.$$

(3) Bending Element with Shear Deformation

When the effect of shear deformation is considered, the deflection u is decomposed into ub and us :

$$u = u^b + u^s$$

where ub is the deflection due to pure bending, and ub is that due to shear, respectively. Noting that equilibrium of shear forces and bending moments in the beam element yields

$$V_i = -V_j = \frac{M_i + M_j}{l} \qquad \left(i.e. \quad M_j = -M_i + V_i l\right).$$

Because of the shearing force, we have

$$\gamma_{zx} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \frac{du^s}{dz} = \frac{\tau_{zx}}{G} = -\frac{V_i}{GA_s} \qquad i.e. \qquad \frac{du^s}{dz} = -\frac{V_i}{GA_s}$$

where G is the shear modulus, As is the effective area of shear deformation defined as a section property of the beam cross section. For example, if the cross section of the beam is rectangular, then $A_s = \frac{5}{6}A$, while it becomes $A_s = \frac{3}{4}A$ if the cross section is circular. Integrating the differential equation, we have

$$u^s{}_i = c \qquad \text{and} \qquad u^s{}_j = c - \frac{V_i l}{GA_s}$$

where c is a constant.

Now noting that the matrix equation of the beam for bending may be written by

$$\boldsymbol{k}_u{}^b \boldsymbol{d}_u{}^b = \boldsymbol{f}_u \qquad \text{that is} \qquad \boldsymbol{k}_u{}^b\left(\boldsymbol{d}_u - \boldsymbol{d}_u{}^s\right) = \boldsymbol{f}_u$$

where

$$\boldsymbol{d}_u{}^b = \begin{Bmatrix} u^b{}_i \\ u^{b'}{}_i \\ u^b{}_j \\ u^{b'}{}_j \end{Bmatrix} \quad , \quad \boldsymbol{d}_u{}^s = \begin{Bmatrix} c \\ 0 \\ c - \dfrac{V_i l}{GA_s} \\ 0 \end{Bmatrix} \quad , \quad \text{and} \quad \boldsymbol{f}_u = \begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix}$$

Here we have used the fact that $u^{s'} = 0$ since the shear deformation is independent of bending moment. Since

$$\boldsymbol{d}_u{}^s = \begin{Bmatrix} c \\ 0 \\ c - \dfrac{V_i l}{GA_s} \\ 0 \end{Bmatrix} = \begin{Bmatrix} c \\ 0 \\ c \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix}$$

we have

$$\boldsymbol{k}_u{}^b \left(\boldsymbol{d}_u - \left\{ \begin{Bmatrix} c \\ 0 \\ c \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} V_i \\ M_i \\ V_j \\ M_j \end{Bmatrix} \right\} \right) = \boldsymbol{f}_u$$

and then

$$\boldsymbol{k}_u{}^b \boldsymbol{d}_u - \boldsymbol{k}_u{}^b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{f}_u = \boldsymbol{f}_u$$

Therefore, we have

$$\boldsymbol{k}_u{}^b \boldsymbol{d}_u = \left(\boldsymbol{I} + \boldsymbol{k}_u{}^b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \boldsymbol{f}_u \quad \Leftrightarrow \quad \left(\boldsymbol{I} + \boldsymbol{k}_u{}^b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1} \boldsymbol{k}_u{}^b \boldsymbol{d}_u = \boldsymbol{f}_u$$

that is, the element stiffness matrix becomes

$$\boldsymbol{k}_u{}^{b+s} = \left(\boldsymbol{I} + \boldsymbol{k}_u{}^b \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{l}{GA_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1} \boldsymbol{k}_u{}^b .$$

Evaluating this by using the matrix obtained in the previous section, we have

$$\boldsymbol{k}_u{}^{b+s} = \frac{EI_{xx}}{l^3(1+\Phi_x)} \begin{bmatrix} 12 & & & sym \\ 6l & (4+\Phi_x)l^2 & & \\ -12 & -6l & 12 & \\ 6l & (2-\Phi_x)l^2 & -6l & (4+\Phi_x)l^2 \end{bmatrix}$$

where

$$\Phi_x = \frac{12EI_{xx}}{GA_{sx}l^2}$$

is the shear deformation parameter such that

$$\Phi_x = \frac{12EI_x}{GA_{sx}l^2} = 24(1+\nu)\frac{A}{A_{sx}}\left(\frac{r_x}{l}\right)^2 \quad \text{with} \quad r_x = \sqrt{\frac{I_x}{A}}$$

, $\frac{r_x}{l}$ is the ratios of radius of gyration to beam element length. If the beam is slender, it becomes zero. Algebra involved in above may be evaluated by, e.g. MATHEMATICA.

The following is a typical algebra by MATHEMATICA to determine the element stiffness matrix including the shear deformation :

```
KB=(EI/L^3)*{{12,6*L,-12,6*L},
             {6*L,4*L^2,-6*L,2*L^2},
             {-12,-6*L,12,-6*L},
             {6*L,2*L^2,-6*L,4*L^2}};
CS=Table[0,{i,1,4},{j,1,4}];
CS[[3,1]]=-L/GAs;
ID=Table[If[i==j,1,0],{i,1,4},{j,1,4}];
KB.CS
IS=Inverse[ID+KB.CS];
KS=Simplify[IS.KB]
```

```
Out[49]=
  12 EI               6 EI               -12 EI
{{------, 0, 0, 0}, {-----, 0, 0, 0}, {------, 0, 0, 0},
       2            GAs L                    2
  GAs L                                 GAs L

   6 EI
  {-----, 0, 0, 0}}
   GAs L

Out[51]=
     12 EI         6 EI         -12 EI        6 EI
{{------------, ----------, ------------, ----------},
  12 EI L    3  12 EI    2  12 EI L    3  12 EI    2
  ------- + L   ----- + L   ------- + L   ----- + L
    GAs          GAs          GAs          GAs

                                       2
     6 EI GAs     4 EI        36 EI           -6 EI GAs
  {-------------, ---- - ---------------, -------------,
              2    L                   3               2
   12 EI + GAs L          12 EI L + GAs L   12 EI + GAs L
```

```
                  2
 2 EI        36 EI
 ---- - ----------------},
  L                     3
        12 EI L + GAs L

    -12 EI GAs       -6 EI GAs        12 EI GAs
{----------------, -------------, ----------------,
                3               2                3
 12 EI L + GAs L   12 EI + GAs L   12 EI L + GAs L

   -6 EI GAs          6 EI GAs
 --------------}, {--------------,
              2               2
 12 EI + GAs L     12 EI + GAs L

                  2
 2 EI        36 EI          -6 EI GAs
 ---- - ----------------, -------------,
  L                     3               2
        12 EI L + GAs L   12 EI + GAs L

                  2
 4 EI        36 EI
 ---- - ----------------}}
  L                     3
        12 EI L + GAs L
```

Similarly, the stiffness matrix of the bending beam with shear deformation about the y axis becomes

$$\boldsymbol{k}_v^{\,b+s} = \frac{EI_{yy}}{l^3\left(1+\Phi_y\right)}\begin{bmatrix} 12 & & & sym \\ -6l & \left(4+\Phi_y\right)l^2 & & \\ -12 & 6l & 12 & \\ -6l & \left(2-\Phi_y\right)l^2 & 6l & \left(4+\Phi_y\right)l^2 \end{bmatrix},$$

where

$$\Phi_y = \frac{12EI_y}{GA_{sy}l^2} = 24(1+\nu)\frac{A}{A_{sy}}\left(\frac{r_y}{l}\right)^2 \quad \text{with} \quad r_y = \sqrt{\frac{I_y}{A}}$$

$\frac{r_y}{l}$ is the ratios of radius of gyration to beam element length. If the beam is slender, it becomes zero.

(4) Saint-Venant Torsion Element

Torsional deformation is governed by

$$U(x,y,z) = -(y - y_s)\theta(z) \quad , \quad V(x,y,z) = +(x - x_s)\theta(z) \quad , \quad W(x,y,z) = \omega_{ns}(x,y)\theta'(z)$$

For the Saint-Venant torsion theory, the angle of twist is assumed to be

$$\theta' = \alpha = \text{constant}$$

in the bar element, and then we assume

$$\theta(z) = c_1 + c_2 z = \{1 \quad z\}\frac{1}{l}\begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}.$$

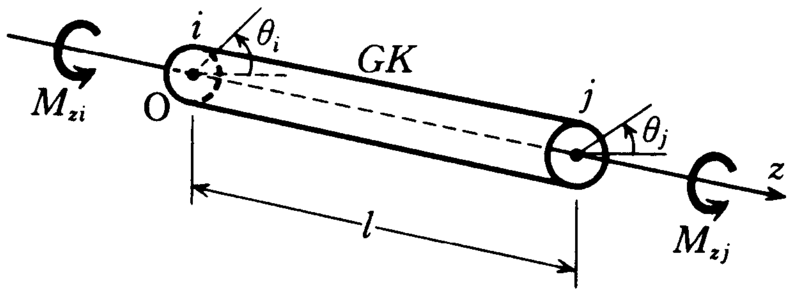


Figure 5 Torsion Bar Element

The shear strains are then obtained by

$$\begin{Bmatrix} \gamma_{zx} \\ \gamma_{zy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial W}{\partial x} + \dfrac{\partial U}{\partial z} \\ \dfrac{\partial W}{\partial y} + \dfrac{\partial V}{\partial z} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \omega_{ns}}{\partial x} - (y - y_s) \\ \dfrac{\partial \omega_{ns}}{\partial y} + (x - x_s) \end{Bmatrix}\theta' = \begin{Bmatrix} \dfrac{\partial \omega_n}{\partial x} - y \\ \dfrac{\partial \omega_n}{\partial y} + x \end{Bmatrix}\theta' = \frac{1}{l}\begin{Bmatrix} \dfrac{\partial \omega_n}{\partial x} - y \\ \dfrac{\partial \omega_n}{\partial y} + x \end{Bmatrix}\{-1 \quad 1\}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

and the strain energy becomes

$$U_e = \frac{1}{2}\int_V (\tau_{zx}\gamma_{zx} + \tau_{zy}\gamma_{zy})dV = \frac{1}{2}\int_V \{\gamma_{zx} \quad \gamma_{zy}\}\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\begin{Bmatrix} \gamma_{zx} \\ \gamma_{zy} \end{Bmatrix}dV$$

$$= \frac{1}{2}\boldsymbol{d}_t^T l\int_A \frac{1}{l}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\left\{\frac{\partial \omega_n}{\partial x} - y \quad \frac{\partial \omega_n}{\partial y} + x\right\}\begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}\frac{1}{l}\begin{Bmatrix} \dfrac{\partial \omega_n}{\partial x} - y \\ \dfrac{\partial \omega_n}{\partial y} + x \end{Bmatrix}\{-1 \quad 1\}dA\boldsymbol{d}_t = \frac{1}{2}\boldsymbol{d}_t^T\boldsymbol{k}_t\boldsymbol{d}_t$$

where

$$\boldsymbol{d}_t = \begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix}$$

$$\boldsymbol{k}_t = \frac{GK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$K = \int_A \left\{ \left(\frac{\partial \omega_n}{\partial x} - y \right)^2 + \left(\frac{\partial \omega_n}{\partial y} + x \right)^2 \right\} dA.$$

The corresponding generalized force vector is defined by the applied torque M_{zi} and M_{zj} at the two end points, i.e.,

$$\boldsymbol{f}_t = \begin{Bmatrix} M_{zi} \\ M_{zj} \end{Bmatrix}$$

and the matrix form of the equilibrium becomes

$$\boldsymbol{k}_t \boldsymbol{d}_t = \boldsymbol{f}_t .$$

The value of K for a typical cross sections we can see in many applications are summerized as follows :

a) elliptic cross section with two radii a and b $\qquad K = \dfrac{\pi a^3 b^3}{a^2 + b^2}$

b) rectangular cross section with b and t $\qquad K \approx \dfrac{bt^3}{3}\left(1 - \dfrac{192}{\pi^5 b}\tanh\dfrac{\pi b}{2t}\right)$

c) prismatic cross sections (Saint-Venant) $\qquad K \approx \dfrac{0.025A^4}{I_p}$

where A is the cross sectional area and I_p is the polar moment of inertia. This approximation is good enough except for the cross sections having one dimension which is much larger than the rest.

d) thin-walled open sections $\qquad K \approx \dfrac{1}{3}\sum_{i=1}^{n} b_i t_i^3$

where t_i is the thickness of the ith thin-walled open section, b_i is the length of the ith thin-walled open section, and n is the total number of sections

e) thin-walled single-cell tubes $K \approx \dfrac{4\Omega^2}{\oint ds/t}$

where Ω is the area enclosed by the center line of the tube wall, t is the thickness, and s is the coordinate along the center line of the tube wall

f) closed tube with fins $K \approx \dfrac{4\Omega^2}{\oint ds/t} + \dfrac{1}{3}\sum_{i=1}^{n} b_i t_i^3$

(5) Element Stiffness Matrix of the Beam Element

Combining all the independent components :

axial deformation
bending deformations in the two orthogonal directions with shear effect
torsional deformation

we have obtained the equilibrium in discrete form :

$$\begin{bmatrix} \boldsymbol{k}_w & 0 & 0 & 0 \\ 0 & \boldsymbol{k}_u & 0 & 0 \\ 0 & 0 & \boldsymbol{k}_v & 0 \\ 0 & 0 & 0 & \boldsymbol{k}_t \end{bmatrix} \begin{Bmatrix} \boldsymbol{d}_w \\ \boldsymbol{d}_u \\ \boldsymbol{d}_v \\ \boldsymbol{d}_t \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f}_w \\ \boldsymbol{f}_u \\ \boldsymbol{f}_v \\ \boldsymbol{f}_t \end{Bmatrix}.$$

Since this form is not convenient in coordinate transformation that is required at the assembling of all the beam elements of a structure, we shall define the element generalized displacement and force vectors as follows :

$$\boldsymbol{d}^T = \{u_i \quad v_i \quad w_i \quad -v_s{'}_i \quad u_s{'}_i \quad \theta_i \quad u_j \quad v_j \quad w_j \quad -v_s{'}_j \quad u_s{'}_j \quad \theta_j\}$$

and

$$\boldsymbol{f}^T = \{V_{xi} \quad V_{yi} \quad P_i \quad M_{xi} \quad M_{yi} \quad M_{zi} \quad V_{xj} \quad V_{yj} \quad P_j \quad M_{xj} \quad M_{yj} \quad M_{zi}\}.$$

Then the element stiffness matrix $\boldsymbol{k}$ becomes

$$
\boldsymbol{k} = \begin{bmatrix}
\frac{12EI_{xx}}{(1+\Phi_x)l^3} & & & & & & & & & & & \\
0 & \frac{12EI_{yy}}{(1+\Phi_y)l^3} & & & & & & & & & & sym \\
0 & 0 & \frac{EA}{l} & & & & & & & & & \\
0 & -\frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(4+\Phi_y)EI_{yy}}{(1+\Phi_y)l} & & & & & & & & \\
\frac{6EI_{xx}}{(1+\Phi_x)l^2} & 0 & 0 & 0 & \frac{(4+\Phi_x)EI_{xx}}{(1+\Phi_x)l} & & & & & & & \\
0 & 0 & 0 & 0 & 0 & \frac{GK}{l} & & & & & & \\
-\frac{12EI_{xx}}{(1+\Phi_x)l^3} & 0 & 0 & 0 & -\frac{6EI_{xx}}{(1+\Phi_x)l^2} & 0 & \frac{12EI_{xx}}{(1+\Phi_x)l^3} & & & & & \\
0 & -\frac{12EI_{yy}}{(1+\Phi_y)l^3} & 0 & \frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & 0 & 0 & \frac{12EI_{yy}}{(1+\Phi_y)l^3} & & & & \\
0 & 0 & -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l} & & & \\
0 & -\frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(2-\Phi_y)EI_{yy}}{(1+\Phi_y)l} & 0 & 0 & 0 & \frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(4+\Phi_y)EI_{yy}}{(1+\Phi_y)l} & & \\
\frac{6EI_{xx}}{(1+\Phi_x)l^2} & 0 & 0 & 0 & \frac{(2-\Phi_x)EI_{xx}}{(1+\Phi_x)l} & 0 & -\frac{6EI_{xx}}{(1+\Phi_x)l^2} & 0 & 0 & 0 & \frac{(4+\Phi_x)EI_{xx}}{(1+\Phi_x)l} & \\
0 & 0 & 0 & 0 & 0 & -\frac{GK}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GK}{l}
\end{bmatrix}
$$

Now, if we define the three orthonormal basis vectors of the local coordinate system (x,y,z) and the global coordinate system (X,Y,Z) by

$$\boldsymbol{e}^T = [\boldsymbol{i}_x \quad \boldsymbol{i}_y \quad \boldsymbol{i}_z] \quad \text{and} \quad \boldsymbol{E}^T = [\boldsymbol{i}_X \quad \boldsymbol{i}_Y \quad \boldsymbol{i}_Z]$$

respectively, they are related to the coordinate transformation matrix $\boldsymbol{T}$ by

$$\boldsymbol{e} = \boldsymbol{TE}$$

where

$$\boldsymbol{T} = \begin{bmatrix} l_{xX} & l_{xY} & l_{xZ} \\ l_{yX} & l_{yY} & l_{yZ} \\ l_{zX} & l_{zY} & l_{zZ} \end{bmatrix},$$

$\{l_{xX} \quad l_{xY} \quad l_{xZ}\}$ is the directional cosine of the local coordinate x with respect to the global coordinate system (X,Y,Z), and similarly, $\{l_{yX} \quad l_{yY} \quad l_{yZ}\}$ and $\{l_{zX} \quad l_{zY} \quad l_{zZ}\}$ are the directional cosines of the local coordinates y and z with respect to the global coordinate system (X,Y,Z), respectively. Therefore, the matrix equation in the local coordinate system is transformed into the one in the global coordinate system (X,Y,Z) :

$$\boldsymbol{k}^G \boldsymbol{d}^G = \boldsymbol{f}^G$$

where

$$\boldsymbol{k}^G = \begin{bmatrix} \boldsymbol{T} & 0 & 0 & 0 \\ 0 & \boldsymbol{T} & 0 & 0 \\ 0 & 0 & \boldsymbol{T} & 0 \\ 0 & 0 & 0 & \boldsymbol{T} \end{bmatrix}^T \boldsymbol{k} \begin{bmatrix} \boldsymbol{T} & 0 & 0 & 0 \\ 0 & \boldsymbol{T} & 0 & 0 \\ 0 & 0 & \boldsymbol{T} & 0 \\ 0 & 0 & 0 & \boldsymbol{T} \end{bmatrix}$$

$$\boldsymbol{d}^G = \left\{ u_i^G \quad v_i^G \quad w_i^G \quad \theta_{Xi}^G \quad \theta_{Yi}^G \quad \theta_{Zi}^G \quad u_j^G \quad v_j^G \quad w_j^G \quad \theta_{Xj}^G \quad \theta_{Yj}^G \quad \theta_{Zj}^G \right\}^T$$

and

$$\boldsymbol{f}^G = \left\{ P_{Xi} \quad P_{Yi} \quad P_{Zi} \quad M_{Xi} \quad M_{Yi} \quad M_{Zi} \quad P_{Xj} \quad P_{Yj} \quad P_{Zj} \quad M_{Xj} \quad M_{Yj} \quad M_{Zj} \right\}^T .$$

Finite Element Analysis of Plane Frame Structures

For plane beams, in the local coordinate system (x,y,z) we have elementwise equilibrium relation from the principle of minimum potential energy, or equivalently from the first Castigliano theorem :

$$\boldsymbol{ku} = \boldsymbol{f}$$

where

$$\boldsymbol{k} = \begin{bmatrix} \frac{12EI_{yy}}{(1+\Phi_y)l^3} & & & & & \text{sym} \\ 0 & \frac{EA}{l} & & & & \\ -\frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(4+\Phi_y)EI_{yy}}{(1+\Phi_y)l} & & & \\ -\frac{12EI_{yy}}{(1+\Phi_y)l^3} & 0 & \frac{6EI_{yy}}{(1+\Phi_y)l^2} & \frac{12EI_{yy}}{(1+\Phi_y)l^3} & & \\ 0 & -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & \\ -\frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(2-\Phi_y)EI_{yy}}{(1+\Phi_y)l} & \frac{6EI_{yy}}{(1+\Phi_y)l^2} & 0 & \frac{(4+\Phi_y)EI_{yy}}{(1+\Phi_y)l} \end{bmatrix}$$

$$\boldsymbol{d} = \begin{Bmatrix} v_{si} \\ w_i \\ -v_s{'}_i \\ v_{sj} \\ w_j \\ -v_s{'}_i \end{Bmatrix} \quad \text{and} \quad \boldsymbol{f} = \begin{Bmatrix} V_{yi} \\ P_i \\ M_{xi} \\ V_j \\ P_j \\ M_{xj} \end{Bmatrix}$$

Here v_s is the transverse deflection of the shear center of the beam in the *y* direction, *w* is the axial displacement of the beam that is the average displacement of the cross section in the *z* direction, $-v_s'$ is the rotation (that is the slope of the transverse deflection vs) about the *x* axis that is orthogonal to the *yz* plane in the right hand coordinate system, while V_y is the transverse shear force in the y direction, *P* is the axial force in the z direction, and M_x is the bending moment about the x axis. Similarly, *E* is Young's modulus, *A* is the area of the cross section, I_{yy} is the moment of inertia of the cross section about the *x* axis, i.e. $I_{yy} = \int_A y^2 dA$, Φ_y is the shear constant in the y axis such that

$$\Phi_y = \frac{12EI_{yy}}{GA_{sy}l^2} = 24(1+\nu)\frac{A}{A_{sy}}\left(\frac{r_y}{l}\right)^2 \quad \text{with} \quad r_y = \sqrt{\frac{I_{yy}}{A}},$$

A_{sy} is the effective area of the cross section for the transverse shear in the y direction, ν is Poisson's ratio, r_y is the radius of gyration, and *l* is the beam length.

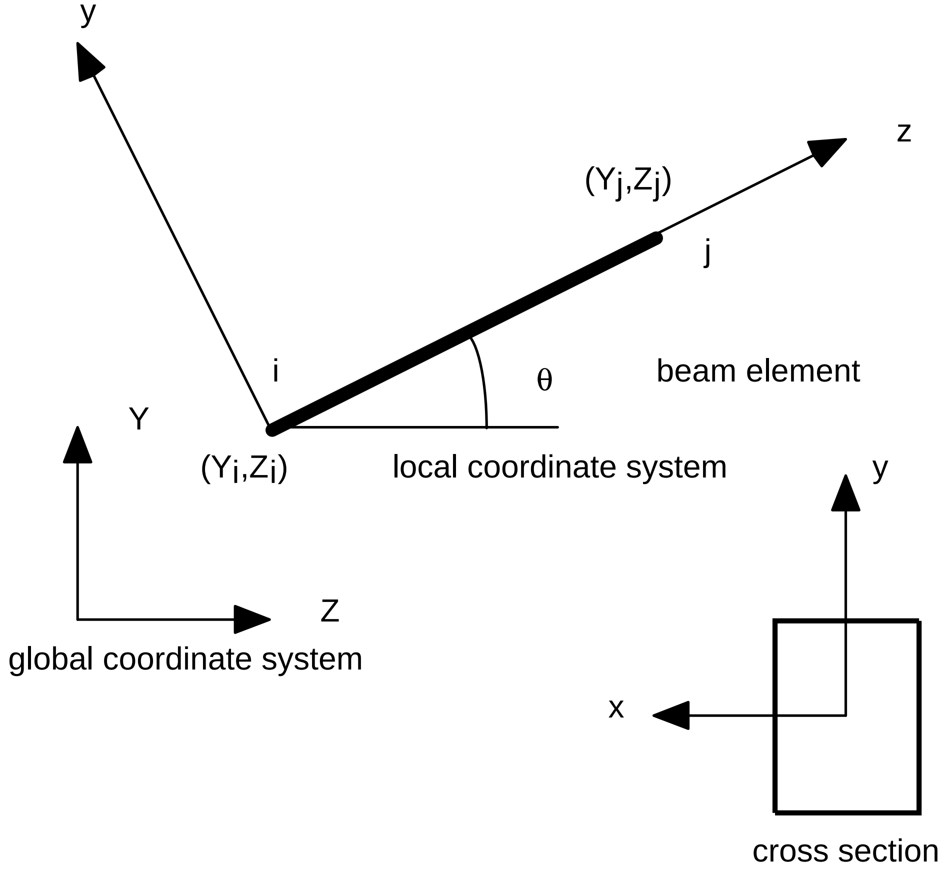


Figure X Global and Local Coordinate Systems of a Plane Beam Element for Side Frame Analysis

Noting that

$$v_s = u_Y \cos\theta - u_Z \sin\theta \quad \text{and} \quad w = u_Y \sin\theta + u_Z \cos\theta$$

we have the transformation matrix

$$\boldsymbol{T} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the relation between the displacements and rotation in the local and global coordinate systems :

$$\begin{Bmatrix} v_s \\ w \\ -v_s' \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_Y \\ u_Z \\ \theta_X \end{Bmatrix} = \boldsymbol{T} \begin{Bmatrix} u_Y \\ u_Z \\ \theta_X \end{Bmatrix}.$$

Defining the transformation matrix for the element stiffness matrix

$$
\boldsymbol{T}^G = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

where

$$
\cos\theta = \frac{Z_j - Z_i}{\sqrt{(Y_j - Y_i)^2 + (Z_j - Z_i)}} \quad \text{and} \quad \sin\theta = \frac{Y_j - Y_i}{\sqrt{(Y_j - Y_i)^2 + (Z_j - Z_i)}}
$$

the element stiffness matrix for the global coodinate system (Y,Z) setting up at the whole structure that can be assembled is given by

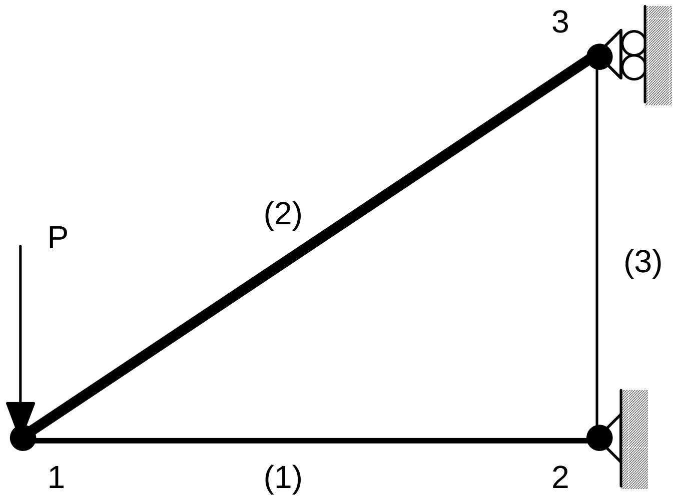
$$
\boldsymbol{k}^G = \boldsymbol{T}^{G^T} \boldsymbol{k} \boldsymbol{T}^G
$$

and the generalized displacement and force vectors for the global coordinate system are defined by

$$
\boldsymbol{d}^G = \begin{Bmatrix} u_{Yi} \\ u_{Zi} \\ \theta_{Xi} \\ u_{Yj} \\ u_{Zj} \\ \theta_{Xj} \end{Bmatrix} \quad \text{and} \quad \boldsymbol{f}^G = \begin{Bmatrix} P_{Yi} \\ P_{Zi} \\ M_{Xi} \\ P_{Yj} \\ P_{Zj} \\ \theta_{Xj} \end{Bmatrix}
$$

where u_Y and u_Z are the displacement in the Y and Z directions, respectively, θ_X is the rotation about the X axis, P_X and P_Y are the forces in the Y and Z direction, respectively, and M_X is the moment about the X axis.

We shall consider an example of a plane beam structure consisting of three beam elements and three nodal points shown in



Here a vertical point force *P* is applied at node 1, while node 2 is fixed in the horizontal and vertical directions and node 3 is supported by hinge roller that can move vertically.

Noting the element connectivities of the structure that describe *i* and *j* nodes of a beam element are given by

element	*i*	*j*
(1)	1	2
(2)	1	3
(3)	2	3

the global stiffness matrix of the whole structure becomes 9 x 9 matrix, while the global generalized displacement and force vectors are 9 component vectors. If the element stiffness matrices for the global coordinate system are expressed by $\boldsymbol{k}^G{}_1$,$\boldsymbol{k}^G{}_2$, and $\boldsymbol{k}^G{}_3$, they are assembled to the global stiffness matrix $\boldsymbol{K}$ for the structure by the following algorithm :

```
for element=1:totalnumberofelement
        % determine the location of the global stiffness matrix of the structure where
        % element stiffness matrix is assembled ( or placed )
        numberofnode=2
        numberofdegreepernode=3
        for nodel=1:numberofnode
                nodeg=ijk(nodel,element)
                for degree=1:numberofdegreepernode
                        il= numberofdegreepernode*(nodel-1)+degree
                        ig=numberofdegreepernode*(nodeg-1)+degree
                        location(il)=ig
                end
        end
        %
        %   form element stiffness matrix ske : 6 x 6 matrix for plane beam elements
```

```
    %
    % assembling of ske to the global one sk
    totaldegree=numberofnode*numberofdegreepernode
    for i=1:totaldegree
        ig=location(i)
        for j=1:totaldegree
            jg=location(j)
            sk(ig,jg)=sk(ig,jg)+ske(i,j)
        end
    end
end
```

Here ske is representating each element stiffness matrix $\boldsymbol{k}^G$ in the global coordinate system, and sk represents the global stiffness matrix $\boldsymbol{K}$ of the whole structure. Array ijk is the list table of the element connectivities. If we perform the above algorithm for assembling to form the global stiffness matrix $\boldsymbol{K}$ of the whole structure, we have

$$\begin{bmatrix} k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(1)}+k_{12}^{(2)} & k_{13}^{(1)}+k_{13}^{(2)} & k_{14}^{(1)} & k_{15}^{(1)} & k_{16}^{(1)} & k_{14}^{(2)} & k_{15}^{(2)} & k_{16}^{(2)} \\ & k_{22}^{(1)}+k_{22}^{(2)} & k_{23}^{(1)}+k_{23}^{(2)} & k_{24}^{(1)} & k_{25}^{(1)} & k_{26}^{(1)} & k_{24}^{(2)} & k_{25}^{(2)} & k_{26}^{(2)} \\ & & k_{33}^{(1)}+k_{33}^{(2)} & k_{34}^{(1)} & k_{35}^{(1)} & k_{36}^{(1)} & k_{34}^{(2)} & k_{35}^{(2)} & k_{36}^{(2)} \\ & & & k_{44}^{(1)}+k_{11}^{(3)} & k_{45}^{(1)}+k_{12}^{(3)} & k_{46}^{(1)}+k_{13}^{(3)} & k_{14}^{(3)} & k_{15}^{(3)} & k_{16}^{(3)} \\ & & & & k_{55}^{(1)}+k_{22}^{(3)} & k_{56}^{(1)}+k_{23}^{(3)} & k_{24}^{(3)} & k_{25}^{(3)} & k_{26}^{(3)} \\ & & & & & k_{66}^{(1)}+k_{33}^{(3)} & k_{34}^{(3)} & k_{35}^{(3)} & k_{36}^{(3)} \\ & & & & & & k_{44}^{(2)}+k_{44}^{(3)} & k_{45}^{(2)}+k_{45}^{(3)} & k_{46}^{(2)}+k_{46}^{(3)} \\ & & & & & & & k_{55}^{(2)}+k_{55}^{(3)} & k_{56}^{(2)}+k_{56}^{(3)} \\ \text{sym} & & & & & & & & k_{66}^{(2)}+k_{66}^{(3)} \end{bmatrix}$$

Here $k_{ij}^{(e)}$ is the ij component of $\boldsymbol{k}^G$ of beam element (e). Since displacement is constrained at node 2 and node 3, the global displacement vector $\boldsymbol{d}$ must be constrained to satisfy

$$d_4 = d_5 = d_7 = 0 \,.$$

Therefore the global stiffness matrix $\boldsymbol{K}$ must be modified by $\boldsymbol{K}^M$, for example,

$$\begin{bmatrix}
k_{11}^{(1)}+k_{11}^{(2)} & k_{12}^{(1)}+k_{12}^{(2)} & k_{13}^{(1)}+k_{13}^{(2)} & 0 & 0 & k_{16}^{(1)} & 0 & k_{15}^{(2)} & k_{16}^{(2)} \\
 & k_{22}^{(1)}+k_{22}^{(2)} & k_{23}^{(1)}+k_{23}^{(2)} & 0 & 0 & k_{26}^{(1)} & 0 & k_{25}^{(2)} & k_{26}^{(2)} \\
 & & k_{33}^{(1)}+k_{33}^{(2)} & 0 & 0 & k_{36}^{(1)} & 0 & k_{35}^{(2)} & k_{36}^{(2)} \\
 & & & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & & k_{66}^{(1)}+k_{33}^{(3)} & 0 & k_{35}^{(3)} & k_{36}^{(3)} \\
 & & & & & & 1 & 0 & 0 \\
 & & & & & & & k_{55}^{(2)}+k_{55}^{(3)} & k_{56}^{(2)}+k_{56}^{(3)} \\
 \text{sym} & & & & & & & & k_{66}^{(2)}+k_{66}^{(3)}
\end{bmatrix}$$

In this modification, we replace the colums and rows related to the degrees of freedom to be constrained by zeros except the unit diagonal terms. The global generalized force $\boldsymbol{f}$ of the whole structure becomes

$$\boldsymbol{f}^T = \{0 \quad -P \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\},$$

and then the equilibrium of the whole structure is written by

$$\boldsymbol{K}^M \boldsymbol{d} = \boldsymbol{f}\ .$$

Since the modified stiffness matrix $\boldsymbol{K}^M$ is not singular, we can solve the matrix equation by

$$\boldsymbol{d} = \left(\boldsymbol{K}^M\right)^{-1} \boldsymbol{f}\ .$$

Once the global generalized displacement d is computed, we have displacement components u_Y and u_Z and the rotation about the X axis at the end points i and j of each beam element, we can transform these into the components in the local coordinate system (x,y,z) so that strains, stresses, and strain energy of each beam element are caluculated in the local coordinate system.

A MATLAB Program for Plane Frame Analysis

We shall develop a MATLAB program for plane beam analysis with possibly flexible joints in order to make finite element (FE) study on side frame analysis and examination of topology of a car body structure. For simplicity, we assume that a structure can be modeled as a plane frame with flexible joints, although most of frame structures in real automotive bodies behave essentially three dimensionally.

A FE program for plane frame structures is written in MATLAB, and only the beam element and flexible joint element are used to model a side frame of a automotive body. The beam element is defined by two end nodes i and j, and three degrees of freedom { u_Y, u_Z, θ_X} are assumed at each node, where u_Y and u_Z are the displacement components in the *Y* and *Z* global coordinate system for the structure, respectively, and θ_X is the rotation about the *X* axis perpendicular to the *Y* and *Z* axes that is the same with the rotation about the *x* axis orthogonal to the local coordinate system *y* and *z*, where the *z* axis is parallel to the beam axis passing through the shear center of the beam cross section. The possibly flexible joint element consists of two nodes *i* and *j* whose nodal coordinates are the same with the three degrees of freedom { u_Y, u_Z, θ_X} which are the same with the beam element, and its stiffness matrix kG in the global coordinate system (Y,Z) is given by

$$\boldsymbol{k}^G = \begin{bmatrix} k_Y & 0 & 0 & -k_Y & 0 & 0 \\ 0 & k_Z & 0 & 0 & -k_Z & 0 \\ 0 & 0 & k_{\theta X} & 0 & 0 & -k_{\theta X} \\ -k_Y & 0 & 0 & k_Y & 0 & 0 \\ 0 & -k_Z & 0 & 0 & k_Z & 0 \\ 0 & 0 & -k_{\theta X} & 0 & 0 & k_{\theta X} \end{bmatrix}$$

where k_Y, k_Z, and $k_{\theta X}$ are the stiffness of the flexible joint in the *Y* and *Z* displacement components, respectively, and of the rotation about the *X* axis. In this sense, we may consider k_Y and k_Z are the spring constants of the springs inserted to the two nodes *i* and *j*, while $k_{\theta X}$ is the rotational spring constant of the torsional spring attached at the two nodes *i* and *j*. In the MATLAB program, we are specifying these stiffness by the percent of the average stiffness of the beam elements connected at the flexible joint.

For simplicity, sequence of the node numbers is made from 1 with no skipping, that is, the node numbers are started from 1 and ended at the total number of nodes *nx*.

The following is the list of the MATLAB program which is more or less self-explained. It is strongly recommended to be modified for your study so that you develop your own analysis tool for side frame analysis and topology consideration. The program consists of three parts : pre-processing, FE analysis, and post-processing.

```
% MEAM 599- 02 / 1998 Winter
% FEM for Side Frame Analysis using plane beam elements
% with possibly flexible joints
%                                                    N. Kikuchi
% ______________________________________________________________
%
% Pre-Processing / set up an analysis model
%
nfile=input('Is data in the datafile ?  [y/n] = ','s');
if nfile~='y'
% read nodal coordinates ( Z, Y )
nx=input('number of nodes of the whole structure = ');
for i=1:nx
       i
       Z(i)=input('Z coordinate of node = ');
       Y(i)=input('Y coordinate of node = ');
```

```
end
% plot the nodes of the plane beam structure
plot(Z,Y,'+')
% read element connectivity and section type of beam elements
%     ijk(1,nel)=node i of beam element nel
%     ijk(2,nel)=node j of beam element nel
%     ijk(3,nel)=section type of beam element nel
nelx=input('number of beam elements = ');
for nel=1:nelx
       nel
       ijk(1,nel)=input('node i = ');
       ijk(2,nel)=input('node j = ');
       ijk(3,nel)=input('section type = ');
end
% plot the beam elements
for nel=1:nelx
       Ze(2*nel-1)=Z(ijk(1,nel));
       Ze(2*nel)=Z(ijk(2,nel));
       Ye(2*nel-1)=Y(ijk(1,nel));
       Ye(2*nel)=Y(ijk(2,nel));
end
plot(Z,Y,'+',Ze,Ye)
% read section type ( properties )
nsecx=input('total number of section type = ');
for i=1:nsecx
       i
       E(i)=input('Young,s modulus / E = ');
       A(i)=input('crossectional area / A = ');
       Iyy(i)=input('moment of inertia about the x axis / Iyy = ');
       Fy(i)=input('shear constant / Fy = ');
end
% read data for flexible joints
nfjx=input('number of flexible joints = ');
if nfjx>0
       for i=1:nfjx
              i
              fjoint(1,i)=input('node i of the flexible joint = ');
              fjoint(2,i)=input('node j of the flexible joint = ');
              kZ(i)=input('stiffness percent in the Z direction / kZ = ');
              kY(i)=input('stiffness percent in the Y direction / kY = ');
              kqX(i)=input('stiffness percent about the X axis rotation / kqX
= ');
       end
end
% plot the flexible joints
if nfjx>0
       for i=1:nfjx
              Zfj(i)=Z(fjoint(1,i));
              Yfj(i)=Y(fjoint(1,i));
       end
plot(Z,Y,Ze,Ye,'+',Zfj,Yfj,'o')
end
% displacement constraints
spc=[];
nspc=input('number of single point constraints = ');
for i=1:nspc
       i
       spc(1,i)=input('node number = ');
       spc(2,i)=input('degree of freedom for spc = ');
       spc(3,i)=input('constrained value = ');
end
% applied forces and moments at the nodes
afm=[];
nafm=input('number of applied forces and moment = ');
```

```
for i=1:nafm
	i
	afm(1,i)=input('node number = ');
	afm(2,i)=input('degrees of freedom = ');
	afm(3,i)=input('applied forces or moment = ');
end
%
save input.mat nx Y Z nelx ijk nsecx E A Iyy Fy nfjx fjoint kY kZ kqX nspc
spc nafm afm
end
if nfile=='y'
	load input.mat
%	load input.mat nx Y Z nelx ijk nsecx E A Iyy Fy nfjx fjoint kY kZ kqX
nspc spc nafm afm
end
%
% FE-Processing / forming the global stiffness matrix
%
% beam elements
%
sk=zeros(3*nx);
f=zeros(3*nx,1);
sksize=3*nx;
%
for nel=1:nelx
%
	Zji=Z(ijk(2,nel))-Z(ijk(1,nel));
	Yji=Y(ijk(2,nel))-Y(ijk(1,nel));
	Lji=sqrt(Zji^2+Yji^2);
	cji=Zji/Lji;
	sji=Yji/Lji;
	TG=zeros(6);
	TG(1,1)= cji;
	TG(1,2)=-sji;
	TG(2,1)= sji;
	TG(2,2)= cji;
	TG(3,3)= 1;
	TG(4,4)= cji;
	TG(4,5)=-sji;
	TG(5,4)= sji;
	TG(5,5)= cji;
	TG(6,6)= 1;
%
	skel=zeros(6);
	ijk3=ijk(3,nel);
	EIyy=E(ijk3)*Iyy(ijk3);
	Fy1=(1+Fy(ijk3));
	skel(1,1)=12*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji^3);
	skel(2,2)=E(ijk3)*A(ijk3)/Lji;
	skel(3,1)=-6*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji^2);
	skel(3,3)=(4+Fy(ijk3))*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji);
	skel(4,1)=-skel(1,1);
	skel(4,3)=-skel(3,1);
	skel(4,4)= skel(1,1);
	skel(5,2)=-skel(2,2);
	skel(5,5)= skel(2,2);
	skel(6,1)= skel(3,1);
	skel(6,3)=(2-Fy(ijk3))*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji);
	skel(6,4)=-skel(3,1);
	skel(6,6)= skel(3,3);
	for i=1:5
		for j=1+1:6
			skel(i,j)=skel(j,i);
		end
```

```
        end
%
        skeg=TG'*skel*TG;
%
        ijk1=ijk(1,nel);
        ijk2=ijk(2,nel);
        ndg=[3*ijk1-2,3*ijk1-1,3*ijk1,3*ijk2-2,3*ijk2-1,3*ijk2];
        for i=1:6
                for j=1:6
                        sk(ndg(i),ndg(j))=sk(ndg(i),ndg(j))+skeg(i,j);
                end
        end
%
end
%
% flexible joint elements
%
if nfjx>0
        for nfj=1:nfjx
                skeg=zeros(6);
                ijk1=fjoint(1,nfj);
                ijk2=fjoint(2,nfj);
                kYb=(sk(3*ijk1-2,3*ijk1-2)+sk(3*ijk2-2,3*ijk2-2))/2;
                kZb=(sk(3*ijk1-1,3*ijk1-1)+sk(3*ijk2-1,3*ijk2-1))/2;
                kqXb=(sk(3*ijk1,3*ijk1)+sk(3*ijk2,3*ijk2))/2;
                refrencestiffness=[nfj,kYb,kZb,kqXb]
                skeg(1,1)= kY(nfj)*kYb;
                skeg(4,4)= kY(nfj)*kYb;
                skeg(1,4)=-kY(nfj)*kYb;
                skeg(4,1)=-kY(nfj)*kYb;
                skeg(2,2)= kZ(nfj)*kZb;
                skeg(5,5)= kZ(nfj)*kZb;
                skeg(2,5)=-kZ(nfj)*kZb;
                skeg(5,2)=-kZ(nfj)*kZb;
                skeg(3,3)= kqX(nfj)*kqXb;
                skeg(6,6)= kqX(nfj)*kqXb;
                skeg(3,6)=-kqX(nfj)*kqXb;
                skeg(6,3)=-kqX(nfj)*kqXb;
                ndg=[3*ijk1-2,3*ijk1-1,3*ijk1,3*ijk2-2,3*ijk2-1,3*ijk2];
                for i=1:6
                        for j=1:6
                                sk(ndg(i),ndg(j))=sk(ndg(i),ndg(j))+skeg(i,j);
                        end
                end
        end
end
%
% single point constraints
%
for i=1:nspc
        spc1=spc(1,i);
        spc2=spc(2,i);
        spc3=spc(3,i);
        ndof=3*(spc1-1)+spc2;
        f=f-sk(:,ndof)*spc3;
        sk(:,ndof)=zeros(sksize,1);
        sk(ndof,:)=zeros(1,sksize);
        sk(ndof,ndof)=1;
end
%
% applied for#es and moments
%
for i=1:nafm
        afm1=afm(1,i);
```

```
        afm2=afm(2,i);
        afm3=afm(3,i);
        ndof=3*(afm1-1)+afm2;
        f(ndof)=f(ndof)+afm3;
end
%
% solving the matrix equation
%
d=sk\f;
d'
%
% Post-Processing of the computed results
%
strainenergy=[];
Zp=[];
Yp=[];
uZ=[];
uY=[];
for nel=1:nelx
        Zji=Z(ijk(2,nel))-Z(ijk(1,nel));
        Yji=Y(ijk(2,nel))-Y(ijk(1,nel));
        Lji=sqrt(Zji^2+Yji^2);
        cji=Zji/Lji;
        sji=Yji/Lji;
        TG=zeros(6);
        TG(1,1)= cji;
        TG(1,2)=-sji;
        TG(2,1)= sji;
        TG(2,2)= cji;
        TG(3,3)= 1;
        TG(4,4)= cji;
        TG(4,5)=-sji;
        TG(5,4)= sji;
        TG(5,5)= cji;
        TG(6,6)= 1;
        deg=zeros(6,1);
        for i=1:2
                ijki=ijk(i,nel);
                for j=1:3
                        deg(3*(i-1)+j)=d(3*(ijki-1)+j);
                end
        end
        del=TG*deg;
        skel=zeros(6);
        ijk3=ijk(3,nel);
        EIyy=E(ijk3)*Iyy(ijk3);
        Fy1=(1+Fy(ijk3));
        skel(1,1)=12*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji^3);
        skel(2,2)=E(ijk3)*A(ijk3)/Lji;
        skel(3,1)=-6*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji^2);
        skel(3,3)=(4+Fy(ijk3))*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji);
        skel(4,1)=-skel(1,1);
        skel(4,3)=-skel(3,1);
        skel(4,4)= skel(1,1);
        skel(5,2)=-skel(2,2);
        skel(5,5)= skel(2,2);
        skel(6,1)= skel(3,1);
        skel(6,3)=(2-Fy(ijk3))*E(ijk3)*Iyy(ijk3)/((1+Fy(ijk3))*Lji);
        skel(6,4)=-skel(3,1);
        skel(6,6)= skel(3,3);
        for i=1:5
                for j=1+1:6
                        skel(i,j)=skel(j,i);
                end
```

```
        end
        nel
        strainenergynel=(del'*skel*del)/2
        strainenergy(nel)=strainenergynel;
% axial strain  ez = ez0 + y ( ezi*(1-z/l) + ezj*(z/l) )
        ez0=(del(5)-del(2))/Lji;
        ezi= 6*(del(1)-del(4))/Lji^2-(4*del(3)+2*del(6))/Lji;
        ezj=-6*(del(1)-del(4))/Lji^2+(2*del(3)+4*del(6))/Lji;
        axialstrain=[ez0,ezi,ezj]
% axial stress  sz = sz0 + y ( szi*(1-z/l) + szj*(z/l) )
        sz0=E(ijk3)*ez0;
        szi=E(ijk3)*ezi;
        szj=E(ijk3)*ezj;
        axialstress=[sz0,szi,szj]
% displacement of the beam axis
        ipx=11;
        for ip=1:ipx
                zpi=(ip-1)/(ipx-1);
                nv=[1-3*zpi^2+2*zpi^3,Lji*(zpi-2*zpi^2+zpi^3),3*zpi^2-
2*zpi^3,Lji*(-zpi^2+zpi^3)];
                nw=[1-zpi,zpi];
                uyi=nv*[del(1),-del(3),del(4),-del(6)]';
                uzi=nw*[del(2),del(5)]';
                ipnel=ipx*(nel-1)+ip;
                Yp(ipnel)=nw*[Y(ijk(1,nel)),Y(ijk(2,nel))]';
                Zp(ipnel)=nw*[Z(ijk(1,nel)),Z(ijk(2,nel))]';
                uY(ipnel)= uyi*cji+uzi*sji;
                uZ(ipnel)=-uyi*sji+uzi*cji;
        end
end
if nfjx>0
        for nfj=1:nfjx
                skeg=zeros(6);
                ijk1=fjoint(1,nfj);
                ijk2=fjoint(2,nfj);
                kYb=(sk(3*ijk1-2,3*ijk1-2)+sk(3*ijk2-2,3*ijk2-2))/2;
                kZb=(sk(3*ijk1-1,3*ijk1-1)+sk(3*ijk2-1,3*ijk2-1))/2;
                kqXb=(sk(3*ijk1,3*ijk1)+sk(3*ijk2,3*ijk2))/2;
                skeg(1,1)= kY(nfj)*kYb;
                skeg(4,4)= kY(nfj)*kYb;
                skeg(1,4)=-kY(nfj)*kYb;
                skeg(4,1)=-kY(nfj)*kYb;
                skeg(2,2)= kZ(nfj)*kZb;
                skeg(5,5)= kZ(nfj)*kZb;
                skeg(2,5)=-kZ(nfj)*kZb;
                skeg(5,2)=-kZ(nfj)*kZb;
                skeg(3,3)= kqX(nfj)*kqXb;
                skeg(6,6)= kqX(nfj)*kqXb;
                skeg(3,6)=-kqX(nfj)*kqXb;
                skeg(6,3)=-kqX(nfj)*kqXb;
                for i=1:2
                        ijki=fjoint(i,nfj);
                        for j=1:3
                                deg(3*(i-1)+j)=d(3*(ijki-1)+j);
                        end
                end
                nfj
                strainenergynfj=(deg'*skeg*deg)/2
                strainenergy(nelx+nfj)=strainenergynfj;
        end
end
bar(strainenergy)
xlabel('beam elements and flexible joints')
ylabel('strain energy')
```

```
title('strain energy distribution')
% plot the deformed configuration
mag=input('magnification factor of the fisplacement = ');
Zup=Zp+mag*uZ;
Yup=Yp+mag*uY;
plot(Zp,Yp,Zup,Yup)
title('deformed configuration')
xlabel('Z')
ylabel('Y')
```

<u>Example 1</u> We shall consider an idealized model of the portion of the joint of the center pillar and the rocker frame as shown in Figure X, whose idealized dimensions are given as in the figure. Assuming standard structural steel, whose Young's modulus is 200 Gpa (i.e. 200 kN/mm^2), we shall consider the idealized 50 mm x 100 mm rectangular cross section of the rocker and center pillar with 1 mm thickness of a thin walled box beam. Then this structure is modeled by 6 beam elements and 1 flexible joint defined by node 3 and 6.

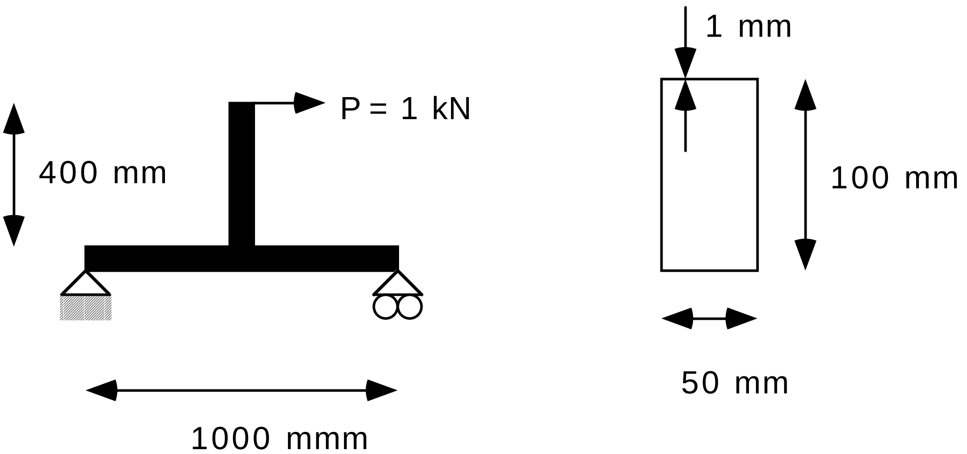


Figure X Schematic View of a Plane Frame Structure

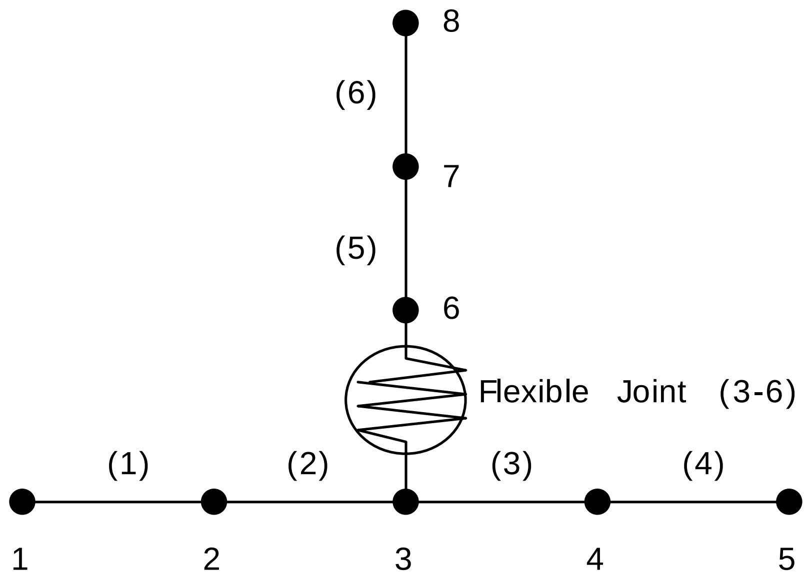
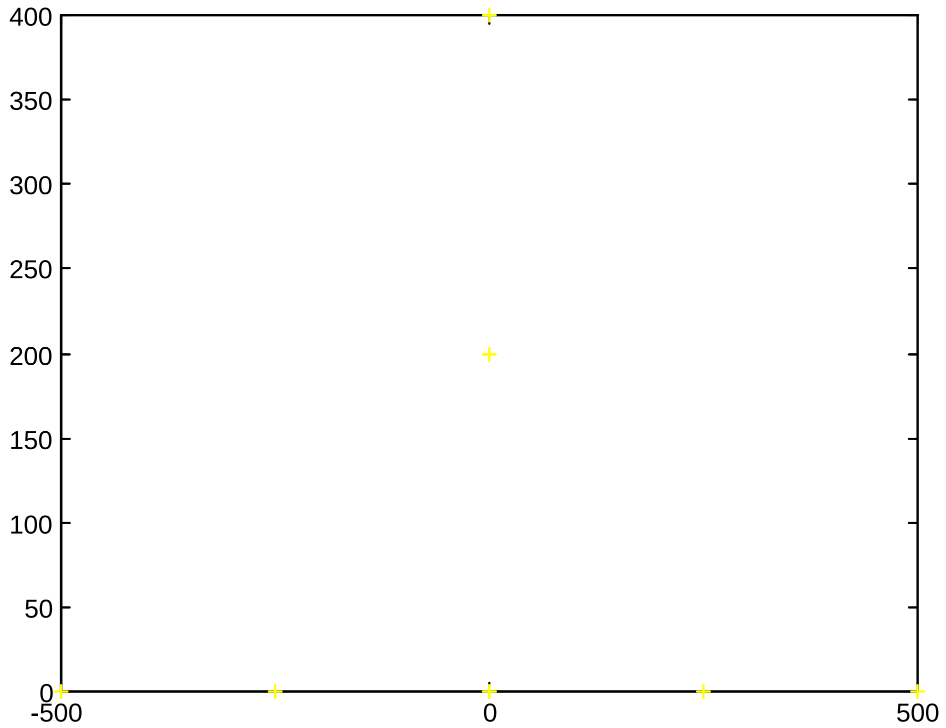


Figure X   A Finite Element Model with a Flexible Joint

Execution of the MATLAB program yields the following result :

```
Is data in the datafile ?  [y/n] = n

number of nodes of the whole structure = 8

i =     1
Z coordinate of node = -500
Y coordinate of node = 0

i =     2
Z coordinate of node = -250
Y coordinate of node = 0

i =     3
Z coordinate of node = 0
Y coordinate of node = 0

i =     4
Z coordinate of node = 250
Y coordinate of node = 0

i =     5
Z coordinate of node = 500
Y coordinate of node = 0

i =     6
Z coordinate of node = 0
Y coordinate of node = 0

i =     7
Z coordinate of node = 0
Y coordinate of node = 200

i =     8
Z coordinate of node = 0
Y coordinate of node = 400
```



number of beam elements = 6

nel = 1
node i = 1
node j = 2
section type = 1

nel = 2
node i = 2
node j = 3
section type = 1

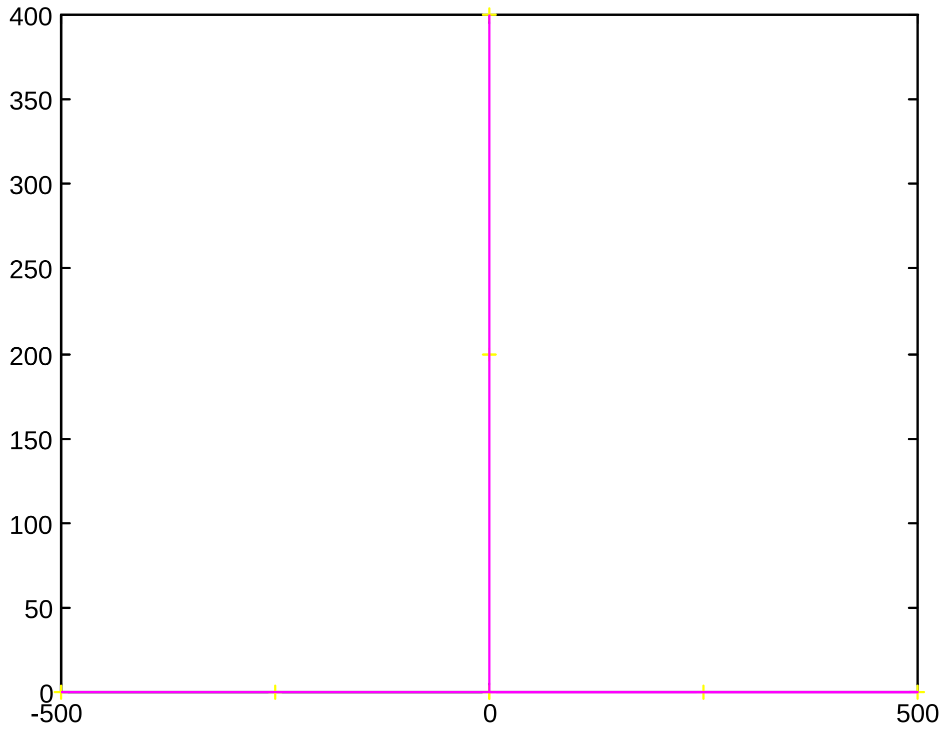
nel = 3
node i = 3
node j = 4
section type = 1

nel = 4
node i = 4
node j = 5
section type = 1

nel = 5
node i = 6

node j = 7
section type = 1

nel = 6
node i = 7
node j = 8
section type = 1



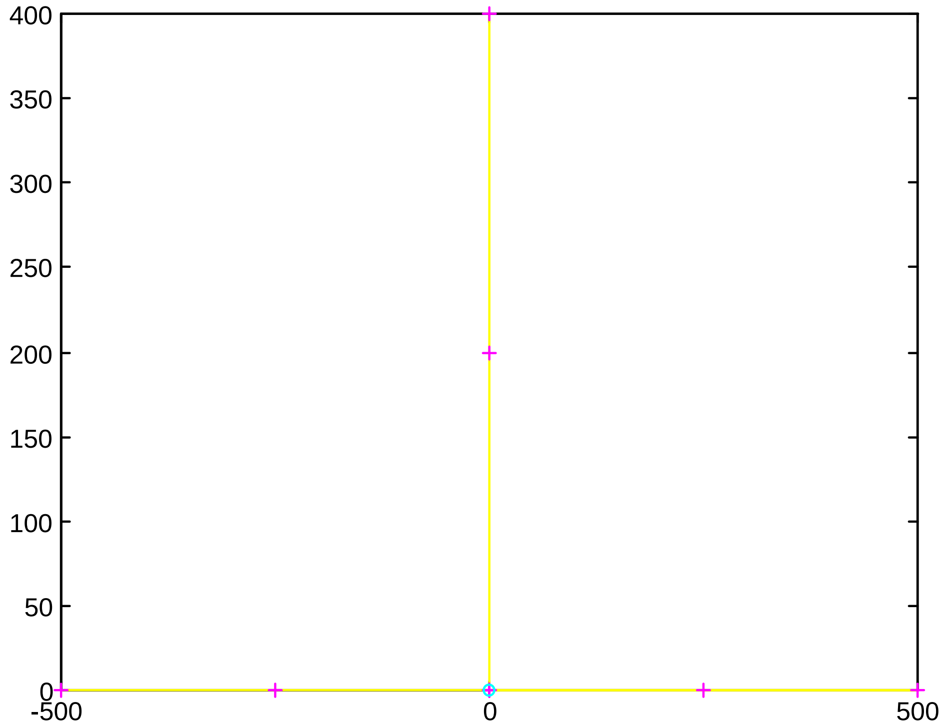
total number of section type = 1

i = 1
Young,s modulus / E = 200
crossectional area / A = 300
moment of inertia about the x axis / Iyy = 401900
shear constant / Fy = 0

number of flexible joints = 1

i = 1
node i of the flexible joint = 3
node j of the flexible j&int = 6
stiffness percent in the Z direction / kZ = 100
stiffness percent in the Y direction / kY = 100

stiffness percent about the X axis rotation / kqX = 0.7



number of single point constraints = 3

i = 1
node number = 1
degree of freedom for spc = 1
constrained value = 0

i = 2
node number = 1
degree of freedom for spc = 2
constrained value = 0

i = 3
node number = 5
degree of freedom for spc = 1
constrained value = 0

number of applied forces and moment = 1

i = 1
node number = 8
degrees of freedom = 2

applied forces or moment = 1

refrencestiffness = 1.0e+06 * 0.0000 0.0002 0.0003 2.0899

ans =

Columns 1 through 7
0 0 -0.0002 0.0389 0.0042 -0.0001 0.0000

Columns 8 through 14
0.0083 0.0004 -0.0389 0.0083 -0.0001 0 0.0083

Columns 15 through 21
-0.0002 0.0000 0.0084 0.0007 0.0000 0.2289 0.0014

Columns 22 through 24
0.0000 0.5490 0.0017

nel = 1
strainenergynel = 0.0073
axialstrain = 1.0e-04 * 0.1667 -0.0000 0.0124
axialstress = 0.0033 -0.0000 0.0002

nel = 2
strainenergynel = 0.0384
axialstrain = 1.0e-04 * 0.1667 0.0124 0.0249
axialstress = 0.0033 0.0002 0.0005

nel = 3
strainenergynel = 0.0363
axialstrain = 1.0e-05 * 0.0000 -0.2488 -0.1244
axialstress = 1.0e-03 * 0.0000 -0.4976 -0.2488

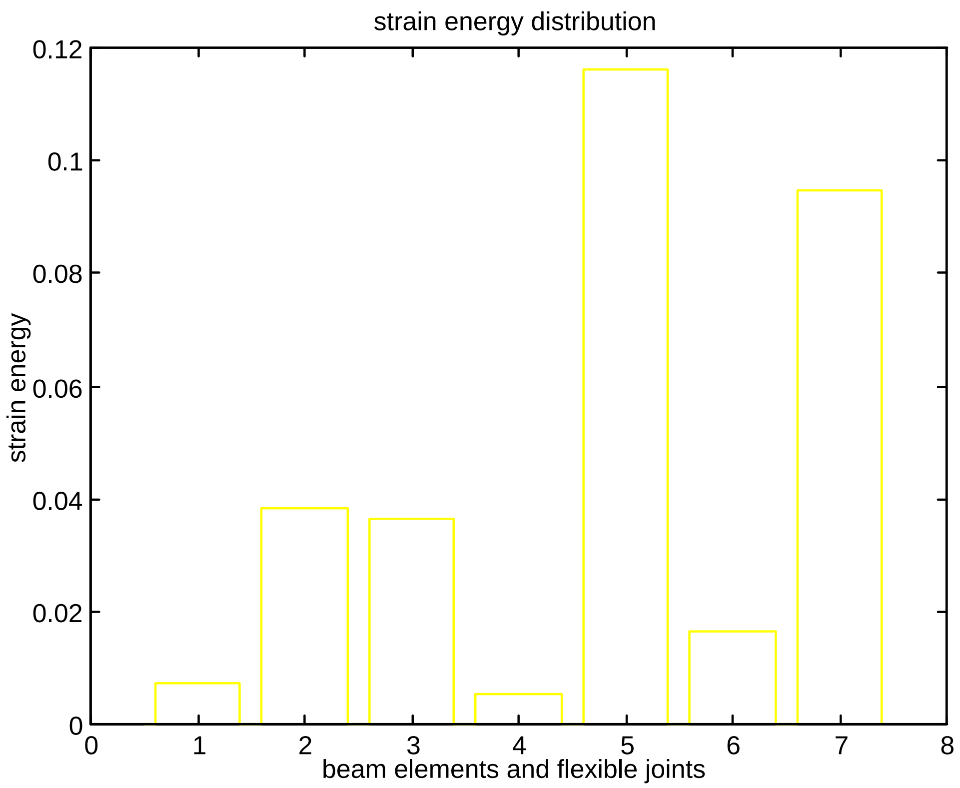
nel = 4
strainenergynel = 0.0052
axialstrain = 1.0e-05 * 0 -0.1244 0.0000
axialstress = 1.0e-03 * 0 -0.2488 0.0000

nel = 5
strainenergynel = 0.1161

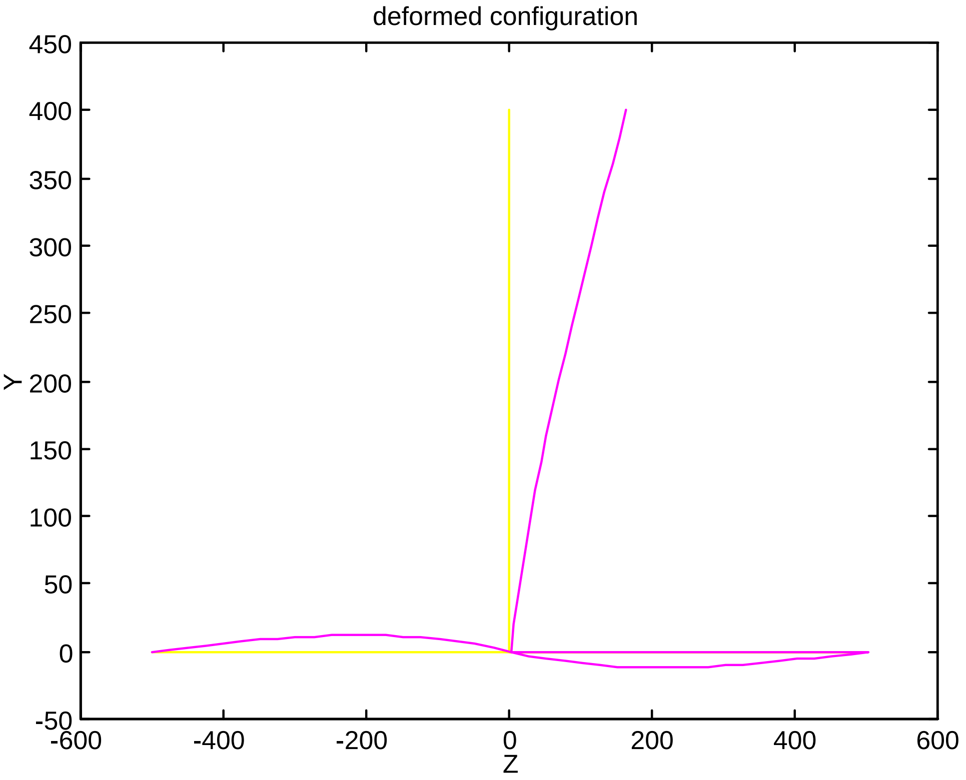
axialstrain = 1.0e-05 * 0 0.4976 0.2488
axialstress = 1.0e-03 * 0 0.9953 0.4976

nel = 6
strainenergynel = 0.0166
axialstrain = 1.0e-05 * 0.0000 0.2488 0
axialstress = 1.0e-03 * 0.0000 0.4976 0

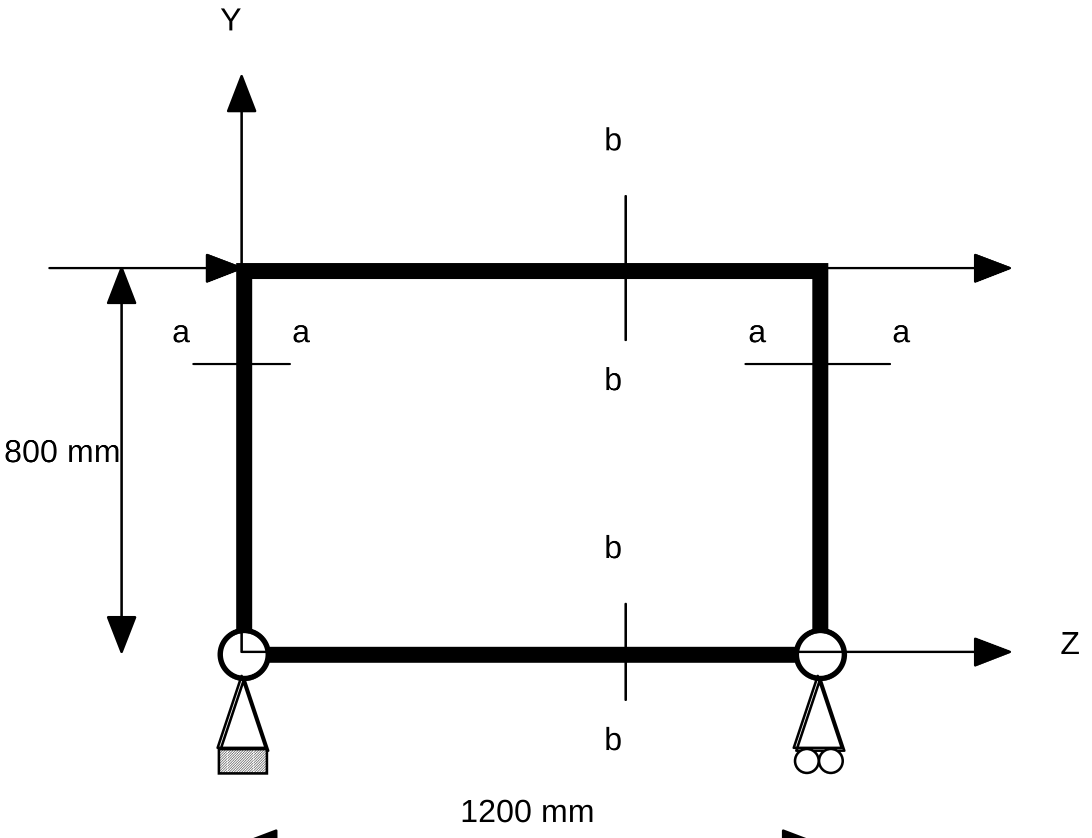
nfj = 1
strainenergynfj = 0.0946



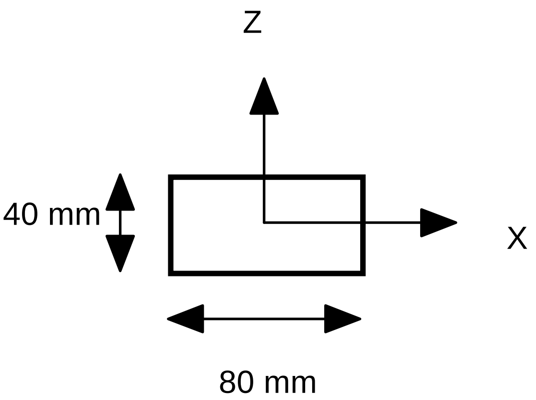
magnification factor of the fisplacement = 300



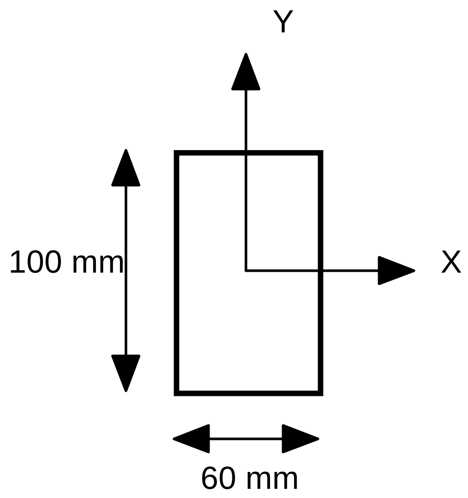
Example 2 We shall consider another plane frame structure consisting of beam and flexible joint elements shown in the following figure :



a - a cross section



b - b cross section



thickness 1 mm

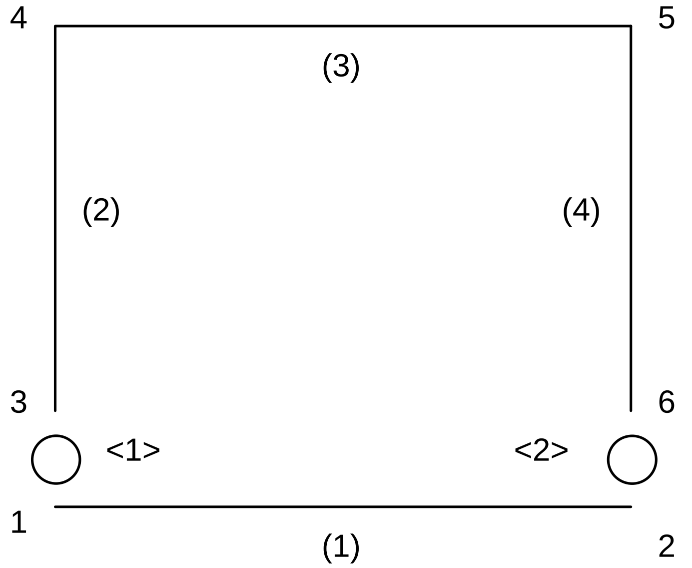
Assuming Young's modulus E = 200 kN/mm2 and zero shear constant Fy = 0, we have

a-a cross section

A = 236 mm2
Iyy = 69999 mmm4

b-b cross section
A = 316 mm2
Iyy = 450910 mm4

Assuming two flexible joints with 70% and 80% rigidity for rotation, while full rigidity is asigned in the axial and transverse displacements, we shall make up a finite element model for the first order analysis by using four beam elements and two flexible joint elements.



For the shear loading at the top beam element, we apply 5 kN horizontal forces at the end points (that is, node 4 and node 5). Input data to the MATLAB program becomes as follows :

Is data in the datafile ? [y/n] = n

number of nodes of the whole structure = 6

i = 1
Z coordinate of node = 0
Y coordinate of node = 0

i = 2
Z coordinate of node = 1200
Y coordinate of node = 0

i = 3
Z coordinate of node = 0

Y coordinate of node = 0

i = 4
Z coordinate of node = 0
Y coordinate of node = 800

i = 5
Z coordinate of node = 1200
Y coordinate of node = 800

i = 6
Z coordinate of node = 1200
Y coordinate of node = 0

number of beam elements = 4

nel = 1
node i = 1
node j = 2
section type = 1

nel = 2
node i = 3
node j = 4
section type = 2

nel = 3
node i = 4
node j = 5
section type = 1

nel = 4
node i = 5
node j = 6
section type = 2

total number of section type = 2

i = 1
Young,s modulus / E = 200
crossectional area / A = 316
moment of inertia about the x axis / Iyy = 450910
shear constant / Fy = 0

i = 2
Young,s modulus / E = 200
crossectional area / A = 236

moment of inertia about the x axis / Iyy = 69999
shear constant / Fy = 0

number of flexible joints = 2

i = 1
node i of the flexible joint = 1
node j of the flexible joint = 3
stiffness percent in the Z direction / kZ = 10
stiffness percent in the Y direction / kY = 10
stiffness percent about the X axis rotation / kqX = 0.7

i = 2
node i of the flexible joint = 2
node j of the flexible joint = 6
stiffness percent in the Z direction / kZ = 10
stiffness percent in the Y direction / kY = 10
stiffness percent about the X axis rotation / kqX = 0.8

number of single point constraints = 3

i = 1
node number = 1
degree of freedom for spc = 1
constrained value = 0

i = 2
node number = 1
degree of freedom for spc = 2
constrained value = 0

i = 3
node number = 2
degree of freedom for spc = 1
constrained value = 0

number of applied forces and moment = 2

i = 1
node number = 4
degrees of freedom = 2
applied forces or moment = 5

i = 2
node number = 5
degrees of freedom = 2
applied forces or moment = 5

refrencestiffness =

1.0e+05 *

0.0000   0.0003   0.0003   1.8530

refrencestiffness =

1.0e+05 *

0.0000   0.0003   0.0003   1.8530

ans =

Columns 1 through 7

0   0   0.0038   0   0.0957   0.0042   0.0123

Columns 8 through 14

0.0187   0.0174   0.0747   24.0790   0.0050   -0.0747   24.0782

Columns 15 through 18

0.0051   -0.0123   0.1148   0.0164

nel =   1
strainenergynel =   7.3671
axialstrain =   1.0e-04 *   0.7979   -0.1958   0.2016
axialstress =   0.0160   -0.0039   0.0040

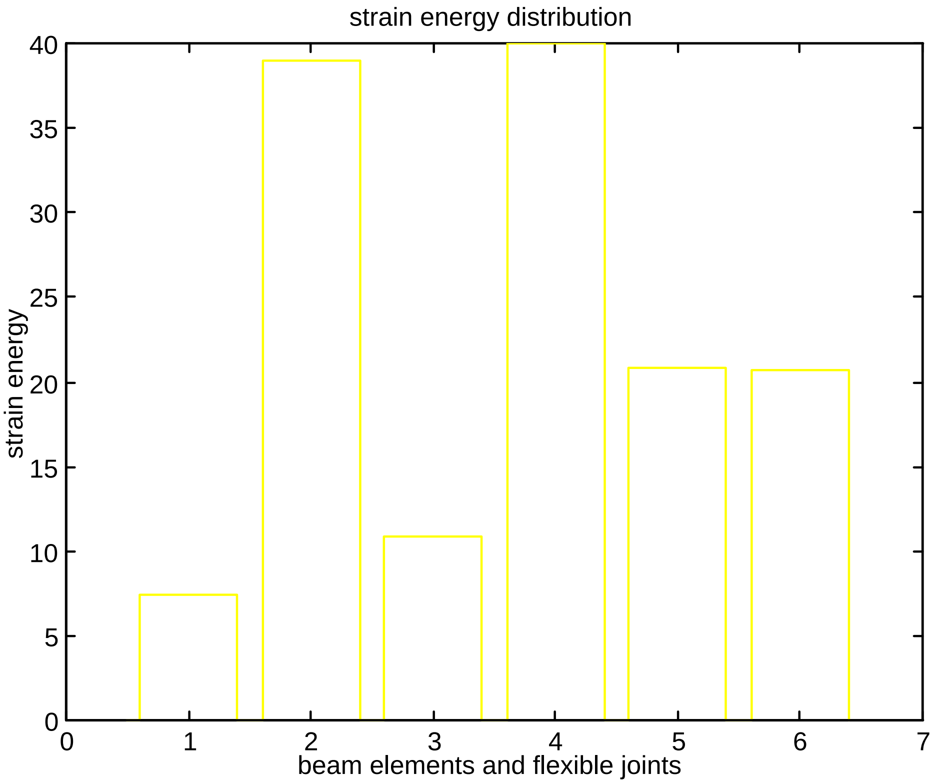
nel =   2
strainenergynel =   38.9183
axialstrain =   1.0e-03 *   0.0780   0.1261   -0.1572
axialstress =   0.0156   0.0252   -0.0314

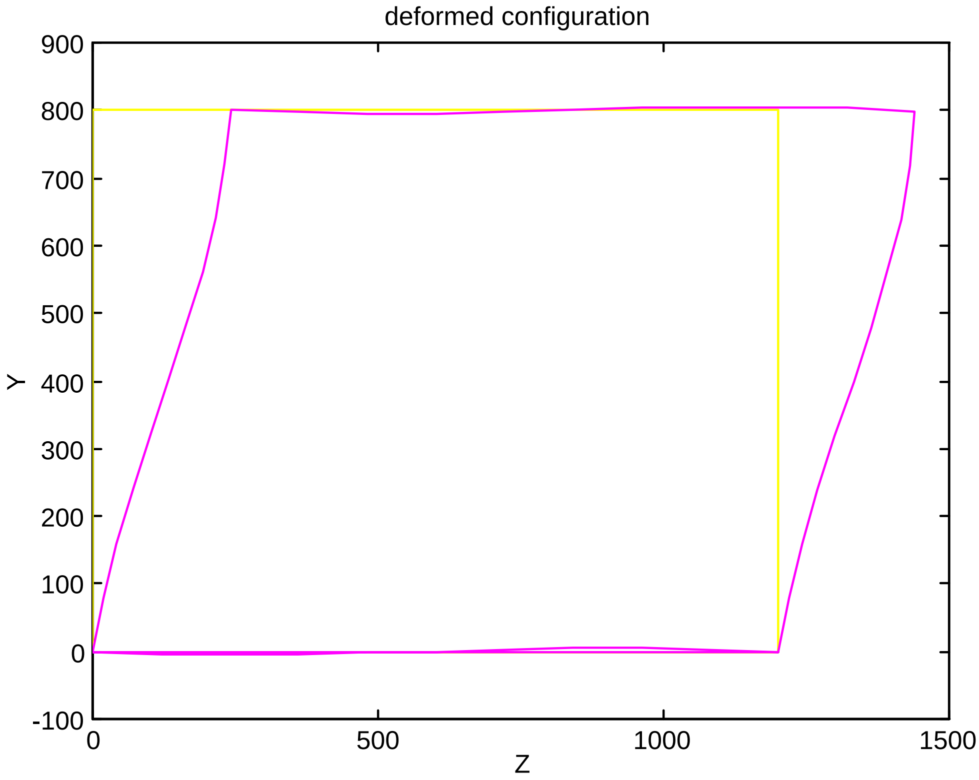
nel =   3
strainenergynel =   10.8133
axialstrain =   1.0e-04 *   -0.0067   -0.2440   0.2457
axialstress =   -0.0001   -0.0049   0.0049

nel =   4
strainenergynel =   39.9879
axialstrain =   1.0e-03 *   -0.0780   0.1583   -0.1299
axialstress =   -0.0156   0.0317   -0.0260

nfj =   1
strainenergynfj =   20.7920

nfj =   2
strainenergynfj =   20.7402





References

1. Gallagher, R.H., Finite Element Analysis : Fundamentals, Prentice-Hall, 1976

2. Martin, H.C., Introduction to Matrix Methods of Structural Analysis, McGraw-Hill, 1966