Final Examination
ME 501 2000 Winter Term
Due Date April 17, 5pm

1. Apply the weighted residual method to the following boundary value problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+k(x)u = f(x) \quad , \quad x \in (0,b) \cup (b,L)$$
$$a(b_-)\frac{du}{dx}(b_-)-a(b_+)\frac{du}{dx}(b_+)=P$$
$$u(0)=u_0$$

, where $g(b_-)=\lim_{\varepsilon\to 0} g(b-\varepsilon)$, $g(b_+)=\lim_{\varepsilon\to 0} g(b+\varepsilon)$, b in a point inside of the interval (0,L), and P is a given number. Using this the weighted residual formulation, approximate it by

$$u(x)=\sum_{j=1}^{n} u_j\phi_j(x) \quad , \quad w(x)=\sum_{i=1}^{n} w_i\psi_i(x)$$

where w is an arbitrary weighting function, and obtain a discrete problem

$$\mathbf{Ku}=\mathbf{f}$$

in terms of a, k, f, P, ϕ_j , ψ_i, and L. That is, find the i-j component of **K** and i component of **f**.

2. (Continuation of Problem 1) Assume $L=a(x)=k(x)=f(x)=1$, $b=\frac{1}{2}$, $P=1$ and $u_0=0$, and

$$\phi_1(x)=\psi_1(x)=\sin\pi x \ , \ \phi_2(x)=\psi_2(x)=\sin 4\pi x$$

(1) Obtain the 2-by-2 matrix **K** and 2 component vector **f**, where $n = m = 2$ are assumed too.
(2) Find the eigenvalues λ of the matrix **K** together with the eigenvectors $\boldsymbol{x}$.
(3) Solve the matrix equation $\mathbf{Ku}=\mathbf{f}$.
(4) Represent the solution **u** as a linear combination of the two eigenvectors of the 2-by-2 matrix **K**.
(5) Find which mode (i.e. eigenvector) is dominant.

3. Answer to the following questions:

(1) For a m-by-n matrix **A**, state the definition of the singular value decomposition of **A**. Find the singular value of the matrix

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & i \\ i & 1 & -i \end{bmatrix} \quad , \quad i=\sqrt{-1}$$

Compute the eigenvalues of $\mathbf{A}^*\mathbf{A}$ and $\mathbf{A}\mathbf{A}^*$, and discuss whether these eigenvalues are related to the singular values obtained.

(2) Using the result of the singular value decomposition, state what is the rank of **A**, what is the range of **A**, and what is the null space of **A**? Find the range and null space of

$$\mathbf{A}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(3) What is an orthogonal matrix **Q**?
(4) When two vectors **u** and **v** are orthogonal?
(5) Show that if a matrix **Q** defined by $\mathbf{Q}=[\mathbf{q}_1,\mathbf{q}_2,.....,\mathbf{q}_n]$ where $\mathbf{q}_i$ are orthonormal, **Q** is an orthogonal matrix.
(6) Describe the Gram-Schmidt orthonormalization process for n number of linearly independent vectors. Orthonormalize the column vectors of the matrix **A** in (2).
(7) What is the definition of linearly independent vectors $\mathbf{v}_i$, $i=1,...,n.$?
(8) What is the Householder transformation **P** defined by a vector **v** ? State the property of **P**. Using the Householder transformation, obtain the QR decomposition of the matrix **A** in Problem (2).
(9) What is the QR algorithm of a matrix **A** to find the eigenvalues? Apply the QR algorithm to find the eigenvalues and eigenvectors of the matrix **A** defined in (2). Further, apply the QR algorithm to find the eigenvalues of

$$\mathbf{B}=\begin{bmatrix} 0 & i & 1 \\ -i & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(10)What is the inverse iteration method? What is the power iteration method? Applying these, find the maximum eigenvalue of **A** in (2).

4. The position vector of an arbitrary point P of a curve C on a two dimensional plane is given by a parametric form $\mathbf{r}(\xi)=\begin{Bmatrix} x(\xi) \\ y(\xi) \end{Bmatrix}$, where ξ is a parametric coordinate in $(0,1)$. Suppose that a coordinate s is defined along the curve, and let s be zero at the point defined by $\xi = 0$, while its value is set as the total length L of the curve at the other end of the curve defined by $\xi = 1$. (a) Establish the relation between s and ξ. (b) How to compute the total length of the curve? (c) How to define the unit tangent vector **t**? (d) What is the unit normal vector **n**? (e) State a way to calculate the curvature?