Midterm Examination
ME501 February 24, 2000

Name__Noboru Kikuchi______________________

1. Consider a boundary value problem

$$-\frac{d^2u}{dx^2} = f \quad in \quad (0, L)$$

with boundary condition

$$u(0) = u(L) = 0,$$

where L is the length of the interval where the differential equation is defined, and f is a given function that is also defined on the interval (0,L).

(1) Find the eigenvalues and eigenfunctions (eigenmodes) of the differential operator $-\frac{d^2}{dx^2}$, that is, find λ and w satisfying the differential equation and the boundary condition:

$$-\frac{d^2w}{dx^2} = \lambda w \quad in \quad (0, L) \quad \& \quad w(0) = w(L) = 0$$

Here w is non-trivial, that is, it is not zero. Hint: w must be trigonometric functions.

Noting that the boundary condition at the both end points is homogeneous, we may assume the solution form $w_k(x) = \sin\left(k\pi \frac{x}{L}\right), k = 1, 2, 3, \ldots.$

Since $\frac{d^2w_k}{dx^2} = -\left(\frac{k\pi}{L}\right)^2 \sin\left(\frac{k\pi}{L}\right) = -\left(\frac{k\pi}{L}\right)^2 w_k$, *we have*

$$-\frac{d^2w_k}{dx^2} = \left(\frac{k\pi}{L}\right)^2 w_k = \lambda_k w_k \quad , \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2$$

that is, the eigenvalue and eigenfunction (λ_k, w_k) *are* $\left(\left(\frac{k\pi}{L}\right)^2, \sin\left(\frac{k\pi}{L}x\right)\right)$.

(2) Noting that there are infinitely many solutions (λ_k, w_k), $k = 1, 2, \ldots..$ in (1), expand the given

function $f(x)$ in terms of the eigenfunctions: $f(x)=\sum_{k=1}^{\infty} f_k w_k(x)$. That is, find the coefficient f_k when the function $f(x)$ is expanded by $f(x)=\sum_{k=1}^{\infty} f_k w_k(x)$.

Assuming the form $f(x)=\sum_{k=1}^{\infty} f_k w_k(x)$, *we have*

$$\int_0^L w_j(x) f(x)dx = \int_0^L w_j(x)\sum_{k=1}^{\infty} f_k w_k(x)dx = \sum_{k=1}^{\infty} f_k \int_0^L w_j w_k dx .$$

Since

$$\begin{aligned}
&\int_0^L w_j w_k dx = \int_0^L \sin\left(\frac{j\pi}{L}x\right)\sin\left(\frac{k\pi}{L}x\right)dx \\
&= -\frac{1}{2}\int_0^L \left\{\cos\left(\frac{j\pi}{L}x+\frac{k\pi}{L}x\right)-\cos\left(\frac{j\pi}{L}x-\frac{k\pi}{L}x\right)\right\}dx \\
&= \frac{1}{2}\left\{\left[\frac{L}{\pi(j+k)}\sin\left(\frac{j\pi}{L}x+\frac{k\pi}{L}x\right)\right]_{x=0}^{x=L} - \left[\frac{L}{\pi(j-k)}\sin\left(\frac{j\pi}{L}x-\frac{k\pi}{L}x\right)\right]_{x=0}^{x=L}\right\} \\
&= 0 \quad \text{if} \quad j \neq k
\end{aligned}$$

$$\begin{aligned}
&\int_0^L w_j w_k dx = \int_0^L \sin\left(\frac{j\pi}{L}x\right)\sin\left(\frac{k\pi}{L}x\right)dx \\
&= -\frac{1}{2}\int_0^L \left\{\cos\left(\frac{j\pi}{L}x+\frac{k\pi}{L}x\right)-1\right\}dx \\
&= \frac{1}{2}\left\{\left[\frac{L}{\pi(j+k)}\sin\left(\frac{j\pi}{L}x+\frac{k\pi}{L}x\right)\right]_{x=0}^{x=L} + \frac{L}{2}\right\} \\
&= \frac{L}{2} \quad \text{if} \quad j = k
\end{aligned}$$

we have

$$\int_0^L w_j(x) f(x)dx = \sum_{k=1}^{\infty} f_k \int_0^L w_j w_k dx = \sum_{k=1}^{\infty} f_k \frac{L}{2}\delta_{kj} = f_j \frac{L}{2}$$

that is

$$f_j = \frac{2}{L}\int_0^L w_j(x)\, f(x)\, dx .$$

(3) Find the solution $u(x)$ of the original boundary value problem by assuming the form $u(x) = \sum_{k=1}^{\infty} u_k w_k(x)$.

Substitution of $u(x) = \sum_{k=1}^{\infty} u_k w_k(x)$ *into the differential equation becomes*

$$-\frac{d^2}{dx^2}\sum_{k=1}^{\infty} u_k w_k(x) = -\sum_{k=1}^{\infty} u_k \frac{d^2 w_k}{dx^2}(x) = \sum_{k=1}^{\infty} u_k \left(\frac{k\pi}{L}\right)^2 w_k(x)$$

$$= f = \sum_{k=1}^{\infty} f_k w_k(x) \quad in \quad (0, L)$$

that is

$u_k = \left(\frac{L}{k\pi}\right)^2 f_k \quad , \quad k = 1, 2, \ldots.$ *Therefore, the solution becomes*

$$u(x) = \sum_{k=1}^{\infty} \left(\frac{L}{k\pi}\right)^2 f_k w_k(x) \quad , \quad f_k = \frac{2}{L}\int_0^L f w_k\, dx$$

It is noted that if k becomes large, contribution of w_k *becomes small.*

2. A data set $\{f_i\} = \begin{Bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$ is given at sampling points $\{x_i\} = \begin{Bmatrix} 0 \\ \pi/4 \\ \pi/2 \end{Bmatrix}$. Assuming the function form $f(x) = \sum_{k=1}^{2} c_k \phi_k(x) = c_1 + c_2 x$, find the coefficients c_1 and c_2 by the least squares method.

Noting that the least squares method with $(n+1)$ *sampling points, can be defined by*

$$\min_{c_k} \frac{1}{2} \sum_{i=1}^{n+1} \left(f_i - \sum_{k=1}^{m+1} c_k \phi_k(x_i) \right)$$

using $(m+1)$ *independent basis functions* $\phi_k(x)$, *we have the minimization problem*

$$\min_{c_1, c_2} F(c_1, c_2)$$

where

$$F(c_1, c_2) = \frac{1}{2}\left(1 - (c_1 + c_2 0)\right)^2 + \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \left(c_1 + c_2 \frac{\pi}{4}\right)\right)^2 + \frac{1}{2}\left(0 - \left(c_1 + c_2 \frac{\pi}{2}\right)\right)^2$$

The necessary condition becomes

$$\frac{\partial F}{\partial c_1} = (c_1 - 1) + \left(c_1 + c_2 \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) + \left(c_1 + c_2 \frac{\pi}{2}\right) = 0$$
$$\frac{\partial F}{\partial c_2} = \frac{\pi}{4}\left(c_1 + c_2 \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) + \frac{\pi}{2}\left(c_1 + c_2 \frac{\pi}{2}\right) = 0$$

that is

$$\begin{bmatrix} 3 & \frac{3\pi}{4} \\ \frac{3\pi}{4} & \frac{5\pi^2}{16} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 1 + \frac{1}{\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{Bmatrix}$$

Solving this yields

$$\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} 3 & \frac{3\pi}{4} \\ \frac{3\pi}{4} & \frac{5\pi^2}{16} \end{bmatrix}^{-1} \begin{Bmatrix} 1+\frac{1}{\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{Bmatrix} = \begin{bmatrix} \frac{5}{6} & -\frac{2}{\pi} \\ -\frac{2}{\pi} & \frac{8}{\pi^2} \end{bmatrix} \begin{Bmatrix} 1+\frac{1}{\sqrt{2}} \\ \frac{\pi}{4\sqrt{2}} \end{Bmatrix} = \begin{Bmatrix} \frac{5+\sqrt{2}}{6} \\ -\frac{2}{\pi} \end{Bmatrix}$$

Therefore, the least squares approximation becomes

$$f\left(x\right) = \frac{5+\sqrt{2}}{6} - \frac{2}{\pi}x$$

3. Let a set of sampling points $\{-1, 0, +1\}$ be given to interpolate a function $f(x) = \cos\left(\frac{\pi}{2}x\right)$ by the Lagrange polynomials. Obtain the Lagrange polynomials at the sampling points, and express the interpolation of $f(x)$ in terms of $f(x) \approx \sum_{i=1}^{3} f(x_i) L_i(x)$ where x_i are the sampling points and $L_i(x)$ are the Lagrange polynomials. What is the value of $\sum_{i=1}^{3} L_i(x)$?

Lagrange polynomials are given by

$$L_1(x) = \frac{x(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)$$

$$L_2(x) = \frac{(x+1)(x-1)}{(0+1)(0-1)} = 1 - x^2$$

$$L_3(x) = \frac{(x+1)x}{(1+1)(1-0)} = \frac{1}{2}x(x+1)$$

Thus, the Lagrange interpolation becomes

$$f(x) \approx f(-1)L_1(x) + f(0)L_2(x) + f(1)L_3(x) = 1 - x^2$$

4. A data set $\{f_i\} = \begin{Bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \end{Bmatrix}$ is given at sampling points $\{x_i\} = \begin{Bmatrix} 0 \\ \pi/4 \\ \pi/2 \end{Bmatrix}$. Assuming the "bell" shape weighting functions $w_i(x)$ by

$$w_1(x) = \begin{cases} 1 - \dfrac{4}{\pi}x \quad , \quad x \in \left(0, \dfrac{\pi}{4}\right) \\ 0 \quad \text{otherwise} \end{cases}$$

$$w_2(x) = \begin{cases} \dfrac{4}{\pi}x \quad , \quad x \in \left(0, \dfrac{\pi}{4}\right) \\ 2 - \dfrac{4}{\pi}x \quad , \quad x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right) \end{cases}$$

$$w_3(x) = \begin{cases} \dfrac{4}{\pi}x - 1 \quad , \quad x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right), \\ 0 \quad \text{otherwise} \end{cases}$$

state the moving least squares method to make curve fit of the given data. Find the solution $a(x)$ in terms of the data $\{f_i\}$ and $\{w_i\}$, and express in the form $a(x) = \sum_{i=1}^{3} f_i \phi_i(x)$. That is, find $\phi_i(x), i = 1,2,3$ in terms of $w_1(x), w_2(x)$, and $w_3(x)$.

Noting that the definition of the moving least squares method is

$$\min_{a(x)} \frac{1}{2} \sum_{i=1}^{n+1} w_i(x)\left(f_i - a(x)\right)$$

for $(n+1)$ *sampling points. The necessary condition of this minimization problem is given by*

$$\frac{\partial}{\partial a}\left\{\frac{1}{2}\sum_{i=1}^{n+1} w_i \left(f_i - a\right)^2\right\} = \left(\sum_{i=1}^{n+1} w_i\right)a - \sum_{i=1}^{n+1} w_i f_i = 0$$

that is

$$a(x) = \frac{\sum_{i=1}^{n+1} w_i f_i}{\sum_{i=1}^{n+1} w_i} = \sum_{i=1}^{n+1} f_i \phi_i(x) \quad , \quad \phi_i(x) = \frac{w_i}{\sum_{j=1}^{n+1} w_j}$$

Since

$$\sum_{i=1}^{3} w_i(x) = 1 \quad in \quad \left(0, \frac{\pi}{2}\right)$$

we have

$$a(x) = \sum_{i=1}^{n+1} f_i w_i(x) = 1w_1(x) + \frac{1}{\sqrt{2}} w_2(x) + 0w_3(x)$$

$$= w_1(x) + \frac{1}{\sqrt{2}} w_2(x)$$

$$= \begin{cases} 1 - \frac{4}{\pi}\left(1 - \frac{1}{\sqrt{2}}\right)x \quad , \quad x \in \left(0, \frac{\pi}{4}\right) \\ \frac{1}{\sqrt{2}}\left(2 - \frac{4}{\pi}x\right) \quad , \quad x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \end{cases}$$

That is, the result is noting but the piecewise linear interpolation.