Review Problems for the Midterm Examination
February 14, 2000

1. Define an inner product of a linear space V that is a set of all continuous functions defined on an interval $(0,1)$, and show that it satisfies the required properties of the inner product. Orthogonalize three functions $\phi_1(x)=1$, $\phi_2(x)=x$, and $\phi_3(x)=x^2$ with respect the inner product you have defined

2. State the required properties of a norm $\|.\|$ in a linear space V. What is the natural norm? Show that the following two norms satisfy the required properties of a norm:

   ① $\|f\| = \max_{x\in[0,1]} |f(x)|$

   ② $\|f\| = \sqrt{\int_0^1 |f(x)|^2 \, dx}$

   where V is a set of all continuous functions defined on $[0,1]$. Show that the inequality

   $$\sqrt{\int_0^1 |f(x)|^2 \, dx} \le \max_{x\in[0,1]} |f(x)|.$$

   State your idea whether or not a positive constant $\alpha > 0$ exists that satisfies the inequality

   $$\alpha \max_{x\in[0,1]} |f(x)| \le \sqrt{\int_0^1 |f(x)|^2 \, dx}.$$

3. Suppose that a data set $\{f_i\}$, $i=1,\ldots,n+1$ is given at a set of sampling points $\{x_i\}$, $i=1,\ldots,n+1$. Define the least squares method to approximate a function $f(x)$ by using a set of linearly independent functions $\{\phi_k(x)\}$, $k=1,\ldots,m+1$. Also find the necessary condition of the least squares method. For the data $\begin{Bmatrix} 0 \\ 1/\sqrt{2} \\ 1 \end{Bmatrix}$ at the sampling points $\begin{Bmatrix} 0 \\ \pi/4 \\ \pi/2 \end{Bmatrix}$, find the least squares solution using $\begin{Bmatrix} \phi_1(x) \\ \phi_2(x) \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$.

4. Define the Lagrange polynomials $L_i^n(x)$ defined on the $(n+1)$ number of sampling points x_i, $i=1,\ldots,n+1$.

5. Define the Legendre polynomials defined on the interval $(-1,1)$. Find the roots of the Legendre

polynomials whose degree of polynomial is 1, 2, and 3.

6. Find the least squares approximation of a function $f(x) = \exp(-x^2)$ by using $\begin{Bmatrix} \phi_1(x) \\ \phi_2(x) \end{Bmatrix} = \begin{Bmatrix} 1 \\ x^2 \end{Bmatrix}$ and an inner product $(.,.)$ defined by

$$(f,g) = \int_{-1}^{1} \{ f(x) g(x) + f'(x) g'(x) \} dx .$$

7. Describe what is the moving least squares method.

Furthermore, review last two homework sets, especially the second one.