5.1

Separating Hyperplanes

Katta G. Murty, IOE 611 Lecture slides

The intersection of family of convex sets is always convex.

The union of two convex sets may not be convex.

The T. of A. Farkas Lemma are separating hyperplane theorems separating a polyhedral cone from a point outside it.

Given nonempty subsets $K_1, K_2 \subset R^n$, a **separating hyperplane** for them is $H = \{x : cx = \alpha\}$ satisfying

$$cx \begin{cases} \geq \alpha & \forall x \in K_1 \\ \leq \alpha & \forall x \in K_2 \end{cases}$$

It is **strict separating hyperplane** if inequalities hold strictly as inequalities.

May not exist for certain pairs. A necessary condition for existence is (Interior of K_1) $\cap$ (interior of K_2) $= \emptyset$. However, even if $K_1 \cap K_2 = \emptyset$, a separating hyperplane may not exist.

Ability to construct separating hyperplanes efficiently

has great significance in LP and in combinatorial optimization.

Sum of two Convex Sets: Given two convex subsets, K_1, K_2 of R^n, their sum, $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$.

The sum of two convex sets is always convex.

THEOREM: $K \subset R^n$ nonempty, closed, convex. $b \in R^n$, $b \notin K$. Then there exists a hyperplane separating b from K.

THEOREM: $K \subset R^n$ convex, nonempty. $b \notin K$. Then K can be separated from b by a hyperplane.

COROLLARY: SUPPORTING HYPERPLANE THEOREM: b a boundary point of a convex set $K \subset R^n$. There exists a hyperplane through b containing K on one of its sides. Such a hyperplane is called a *supporting hyperplane for K at its boundary point b*.

THEOREM: K_1, K_2 convex , nonempty, and $K_1 \cap K_2 = \emptyset$. Then there exists a hyperplane separating K_1 and K_2.

How to find the separating hyperplane?

1. $d \notin K = \{x : A_{i.}x \begin{cases} = b_i, & i = 1 \text{ to } m \\ \geq b_i, & i = m+1 \text{ to } m+p \end{cases} \}$.

2. $b \notin \text{Pos}(A)$.

3. $b \notin K =< A_{.1}, \ldots, A_{.t} >$.

4. $b \notin K = \{x : f_i(x) \leq 0, \ i = 1 \text{ to } m\}$, each $f_i(x)$ is a differentiable convex function.

5. $b \notin K = \{x : g_i(x) \leq 0, \ i = 1 \text{ to } m\}$, each $f_i(x)$ is differentiable but not all of them convex, even though K is known to be convex.

EXAMPLE: M is a P-matrix. $\bar{z} \geq 0$ satisfies $Mz + q \geq 0$ but not all of the following. Separate $\bar{z}$ from set of feasible solutions of following.

$$
\begin{aligned}
-z &\le 0 \\
-Mz - q &\le 0 \\
z^T(Mz + q) &\le 0
\end{aligned}
$$

6. K = convex hull of integer solutions to: $Ax \le d$, each $x_j \in \{0, 1\}$. Given b, efficiently conclude that either $b \in K$ or produce a hyperplane separating b from K.