Chapter 8

POLYNOMIALLY BOUNDED ALGORITHMS FOR SOME CLASSES OF LCPs

In this chapter we discuss algorithms for special classes of LCPs, whose computational complexity is bounded above by a polynomial in either the order or the size of the LCP. We consider the LCP (q, M) where M is either a Z-matrix, or a triangular P-matrix, or an integer PSD-matrix.

8.1 Chandrasekaran's Algorithm for LCPs Associated with Z-Matrices

Consider the LCP (q, M) of order n, where M is a Z-matrix. As discussed in Section 3.4, $M = (m_{ij})$ is a Z-matrix if all its off diagonal entries are nonpositive, that is $m_{ij} \leqq 0$ for all $i \neq j$. The algorithm discussed below by R. Chandrasekaran [8.2], terminates after at most n principal pivot steps, with either a solution of the LCP (q, M) or the conclusion that it has no solution.

The Algorithm

The initial tableau is (8.1)

w	z	
I	$-M$	q

(8.1)

Step 1: Start with the initial tableau and with $w = (w_1, \dots, w_n)$ as the initial complementary basic vector. If this is a feasible basis (i. e., if $q \geqq 0$) it is a complementary feasible basis, terminate. Otherwise, go to the next step.

General Step: Let $\bar{q}$ be the present update right hand side constants vector. If $\bar{q} \geqq 0$, the present basic vector is a complementary feasible basic vector, terminate. Otherwise select a t such that $\bar{q}_t < 0$. Let $-\overline{m}_{tt}$ be the present update entry in the t^{th} row and the column vector of z_t. At this stage, the present basic variable in row t will be w_t (this follows from statement 5 listed below). If $-\overline{m}_{tt} \geqq 0$, there exists no nonnegative solution for (8.1) and consequently the LCP (q, M) has no solution, terminate. Otherwise if $-\overline{m}_{tt} < 0$, perform a principal pivot step in position t and go to the next step.

Using the fact that the initial matrix M is a Z-matrix, we verify that in the initial system (8.1), for any $t = 1$ to n, all the entries in row t are nonnegative with the exception of the entry in the column of z_t. From the manner in which the algorithm is carried out, the following facts can be verified to hold.

1. All pivot elements encountered during the algorithm are strictly negative.
2. For any t such that no pivot step has been performed in the algorithm so far in row t, all the entries in this row on the left hand portion of the present updated tableau are nonnegative, except, possibly the entry in the column of z_t. The infeasibility conclusion in the algorithm follows directly from this fact.
3. If s is such that a pivot step has been carried out in row s in the algorithm, in all subsequent steps, the updated entry in this row in the column of any nonbasic z_i is nonpositive.
4. Once a pivot step has been performed in a row, the updated right hand side constant in it remains nonnegative in all subsequent steps. This follows from statements 1 and 3.
5. Once a variable z_t is made a basic variable, it stays as a basic variable, and its value remains nonnegative in the solution, in all subsequent steps.
6. All basic vectors obtained in the algorithm are complementary, and the algorithm terminates either with the conclusion of infeasibility or with a complementary feasible basis.
7. At most one principal pivot step is carried out in each position, thus the algorithm terminates after at most n pivot steps. Thus the computational effort measured in terms of basic operations like multiplications, additions, comparisons of real numbers, is at most $\mathcal{O}(n^{\mathbf{3}})$.

From these facts we conclude that if the system "$w - Mz = q$, $w \geqq 0$, $z \geqq 0$" is feasible and M is a Z-matrix, then the LCP (q, M) has a complementary feasible solution and the above algorithm finds it. Hence, when M is a Z-matrix, the LCP (q, M) has a solution iff $q \in \text{Pos}(I \,\vdots\, -M)$, or equivalently, every Z-matrix is a Q_0-matrix.

R. W. Cottle and R. S. Sacher, and J. S. Pang [8.7, 8.8] discuss several large scale applications of the LCP basid on this algorithm.

Exercises

8.1 Solve the LCP with the following data by Chandrasekaran's algorithm.

$$M = \begin{pmatrix} 1 & -2 & 0 & -2 & -1 \\ -1 & 0 & -1 & -2 & 0 \\ -2 & -3 & 3 & 0 & 0 \\ 0 & -1 & -1 & -2 & -1 \\ -2 & 0 & -1 & -2 & 3 \end{pmatrix}, \qquad q = \begin{pmatrix} -4 \\ -4 \\ -2 \\ -1 \\ -2 \end{pmatrix}.$$

8.2 Is the complementary pivot method guaranteed to process the LCP (q, M) when M is a Z-matrix ?

8.3 Discuss an efficient method for computing all the complementary solutions of the LCP (q, M) when M is a Z-matrix.

8.2 A Back Substitution Method for the LCPs Associated with Triangular P-Matrices

A square matrix $M = (m_{ij})$ of order n is said to be a lower triangular matrix if $m_{ij} = 0$ for all $j \geqq i+1$. It is upper triangular if M^T is lower triangular. The square matrix M is said to be a triangular matrix if there exists a permutation of its rows and columns which makes it lower triangular. A triangular matrix satisfies the following properties.

(i) The matrix has a row that contains a single nonzero entry.

(ii) The submatrix obtained from the matrix by striking off the row containing a single nonzero entry and the column in which that nonzero entry lies, also satisfies property (i). The same process can be repeated until all the rows and columns of the matrix are struck off.

A lower triangular or an upper triangular matrix is a P-matrix iff all its diagonal entries are strictly positive. A triangular matrix is a P-matrix iff every one of its single nonzero entries identified in the process (i), (ii) above is the diagonal entry in its row and is strictly positive. Thus a triangular matrix is a P-matrix iff there exists a permutation matrix Q such that $Q^T M Q$ is a lower triangular matrix with positive diagonal entries.

Example 8.1

Let

$$M = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Verify that $Q^T M Q = \widetilde{M}(4)$ defined in equation (1.15) for $n = 4$, and hence M is a triangular P-matrix.

If M is a triangular P-matrix, the LCP (q, M) can be solved by the following back substitution method.

Identify the row in $M = (m_{ij})$ containing a single nonzero entry. Suppose it is row t. If $q_t \geqq 0$, make $w_t = q_t$, $z_t = 0 = \bar{z}_t$. On the other hand, if $q_t < 0$, make $w_t = 0$, $z_t = \frac{q_t}{-m_{tt}} = \bar{z}_t$. Add $\bar{z}_t M_{.t}$ to the right hand side constants vector q in (8.1), and then eliminate the columns of w_t, z_t and the t^{th} row from (8.1), thus converting (8.1) into a system of the same form in the remaining variables, on which the same process is repeated.

In this method, the value of one complementary pair of variables (w_i, z_i) are computed in each step, their values are substituted in the other constraints and the process repeated. The method finds the complete solution in n steps.

Example 8.2

Consider the LCP (q, M) with

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -8 \\ -12 \\ -14 \end{pmatrix}.$$

It can be verified that this method leads to the values $(w_1, z_1) = (0, 8)$, $(w_2, z_2) = (4, 0)$, $(w_3, z_3) = (2, 0)$ in that order, yielding the solution $(w_1, w_2, w_3; z_1, z_2, z_3) = (0, 4, 2; 8, 0, 0)$. The same problem was solved by the complementary pivot algorithm in Example 2.10.

8.3 Polynomially Bounded Ellipsoid Algorithms for LCPs Corresponding to Convex Quadratic Programs

In the following sections we show that the ellipsoid algorithms for linear inequalities and LPs (see references [8.13], [2.26]) can be extended to solve LCPs associated with PSD matrices with integer data, in polynomial time. As shown in Chapter 1 every convex quadratic programming problem can be transformed into an LCP associated with a PSD matrix, and hence the methods described here provide polynomially bounded algorithms for solving convex quadratic programs with integer data. These algorithms are taken from S. J. Chung and K. G. Murty [8.4]. Similar work also appeared in [8.14, 8.1] among other references. If the data in the problem is not integer but rational, it could be converted into an equivalent problem with integer data by multiplying all the data by a suitably selected positive integer, and solved by the algorithms discussed here in polynomial time.

In Sections 8.1, 8.2 we discussed algorithm for special classes of LCPs in which the computational effort required to solve an LCP of order n is at most $\mathcal{O}(n^{\mathbf{3}})$. These algorithms do not require the data in the problem to be integer or rational, it could even be irrational as long as the matrix M satisfies the property of being a Z-matrix or triangular P-matrix as specified and the required arithmetical operations can be carried out on the data with the desired degree of precision. Thus these algorithms discussed in Section 8.1, 8.2 are extremely efficient and practically useful to solve LCPs of the types discussed there. The ellipsoid algorithms discussed in the following sections have an entirely different character. They are polynomially bounded as long as M is an integer PSD-matrix, but their computational complexity is not bounded above by a polynomial in the order of the problem, but by a polynomial in the size of the problem (the size of the problem is the total number of digits in all the data when it is encoded using binary encoding). From Chapter 6 we know that in the worst case, the complementary and principal pivoting method discussed earlier are not polynomially bounded. However, in computational tests on practical, or randomly generated problems, the observed average computational effort required by ellipsoid method turned out to be far in excess of that required by complementary and principal pivoting methods. Also, in the ellipsoid methods, each computation has to be carried out to a large number of digits of precision, making it very hard to implement them on existing computers.

Thus the ellipsoid algorithms discussed in the following sections are not likely to be practically useful, at least not in their present forms. The major importance of these ellipsoid methods is theoretical, they made it possible for us to prove that convex quadratic programs, or equivalently LCPs associated with PSD-matrices with integer data, are polynomially solvable.

Size of an LCP

In this and in subsequent sections, we use the symbol L to denote the size of the problem istance, it is the total number of binary digits in all the data in the instance, assuming that all the data is integer. Given an integer α, the total number of binary digits in it (i. e., the number of bits needed to encode it in binary form) is approximately $\lceil 1+\log_2(1+|\alpha|)\rceil$, the celing of $\big(1+\log_2(1+|\alpha|)\big)$, that is, the positive integer just $\geqq \big(1+\log_2(1+|\alpha|)\big)$. Since the data in an LCP (q, M) of order n is n, q, M, we can define the size of this LCP to be

$$L = \left\lceil (1 + \log_2 n) + \sum_{i,j=1}^{n} \Big(1 + \log_2(1 + |m_{ij}|)\Big) + \sum_{j=1}^{n} \Big(1 + \log_2(1 + |q_j|)\Big) \right\rceil .$$

An Ellipsoid in $\mathbf{R}^n$

An ellipsoid in $\mathbf{R}^n$ is uniquely specified by its center $p \in \mathbf{R}^n$ and a positive definite matrix D of order n. Given these, the ellipsoid corresponding to them is $\{x : (x - p)^T D^{-1}(x - p) \leqq 1\}$ and is denoted by $\mathbf{E}(p, D)$. Notice that if $D = I$, the ellipsoid $\mathbf{E}(p, D)$ is the solid spherical ball with p as center and the radius equal to 1. When D is positive definite, for $x, y \in \mathbf{R}^n$, the function $f(x, y) = (x - y)^T D^{-1}(x - y)$ is called the **distance between x and y with D^{-1} as the metric matrix** (if $D = I$, this becomes the usual **Euclidean distance**). The ellipsoid methods discussed in the following sections obtain a new ellipsoid in each step by changing the metric matrix. Hence these methods belong to the family of **variable metric methods**. Also, the formula for updating the metric matrix from step to step is of the form D_{r+1} = a constant times $(D_r + C_r)$, where D_j is the metric matrix in step j for $j = r, r+1$; and C_r is a square matrix of order n and rank 1 obtained by multiplying a column vector in $\mathbf{R}^n$ by its transpose. Methods which update the metric matrix by such a formula are called **rank one methods** in nonlinear programming literature. Rank one methods and variables metric methods are used extensively for solving convex unconstrained minimization problems in nonlinear programming. See references [10.2, 10.3, 10.9, 10.13]. The ellipsoid methods discussed in the following sections belong to these families of methods.

8.4 An Ellipsoid Algorithm for the Nearest Point Problem on Simplicial Cones

Let $B = (b_{ij})$ be a nonsingular square matrix of order n, and $b = (b_i)$ a column vector in $\mathbf{R}^n$. We assume that all the data in B, b is integer, and consider the nearest point problem $[B; b]$ discussed in Chapter 7. This is equivalent to the LCP $(\bar{q}, \overline{M})$ where $\overline{M} = B^T B$, $\bar{q} = -B^T b$, and so $\overline{M}$, $\bar{q}$ are integer matrices too, and $\overline{M}$ is PD and symmetric. If $b \in \text{Pos}(B)$, then the point b is itself the solution of $[B; b]$, and $(\bar{w} = 0, \bar{z} = B^{-1}b)$ is the unique solution of the LCP $(\bar{q}, \overline{M})$. So we assume that $b \notin \text{Pos}(B)$ (this implies that $b \neq 0$). Here we present an ellipsoid algorithm for solving this nearest point problem $[B; b]$ and the corresponding LCP $(\bar{q}, \overline{M})$. We begin with some results necessary to develop the algorithm.

Definitions

Let ε be a small positive number. Later on we specify how small ε should be. Let

$$\begin{aligned}
\mathbf{K} &= \{x :\ B^{-1}x \geqq 0, B^T(x-b) \geqq 0\} \\
\mathbf{E} &= \{x :\ (x-\tfrac{b}{2})^T(x-\tfrac{b}{2}) \leqq \tfrac{b^Tb}{4}\} \\
\mathrm{Bd}(\mathbf{E}) &= \text{Boundary of } \mathbf{E} = \{x :\ (x-\tfrac{b}{2})^T(x-\tfrac{b}{2}) = \tfrac{b^Tb}{4}\} \\
\mathbf{E}_1 &= \left\{x : (x-\tfrac{b}{2})^T(x-\tfrac{b}{2}) \leqq \left(\varepsilon + \sqrt{\tfrac{b^Tb}{4}}\right)^{\mathbf{2}}\right\} \\
L_1 &= \left\lceil (1+\log_2 n) + \sum_{i,j=1}^{n} \left(1+\log_2(|b_{ij}|+1)\right) + \sum_{i=1}^{n}\left(1+\log_2(|b_i|+1)\right) \right\rceil \\
L_2 &= n(n+1)(L_1+1) \\
L_3 &= \left(n(2n+1)+1\right)L_1 \\
\bar{x} &= \text{Nearest point in } \mathrm{Pos}(B) \text{ to } b \\
\overline{M} &= (\overline{m}_{ij}) = B^TB \\
\bar{q} &= (\bar{q}_i) = -B^Tb \\
\bar{z} &= B^{-1}\bar{x} \\
\bar{w} &= \bar{q} + \overline{M}\bar{z} \\
\delta &= \tfrac{3}{4}2^{-\mathbf{L_2}}.
\end{aligned}$$

Some Preliminary Results

Our nearest point problem $[B; b]$ is equivalent to the LCP $(\bar{q}, \overline{M})$. Each $\overline{m}_{ij}$ or $\bar{q}_i$ is of the form $\gamma_1\gamma_2 + \gamma_3\gamma_4 + \ldots + \gamma_{2n-1}\gamma_{2n}$, where the γ's are entries from B, b, and hence are integer. So we have

$$\begin{aligned}
\log_2|m_{ij}| &= \log_2(|\gamma_1\gamma_2 + \ldots + \gamma_{2n-1}\gamma_{2n}|) \\
&< \log_2\left((|\gamma_1|+2)(|\gamma_2|+2) + \ldots + (|\gamma_{2n-1}|+2)(|\gamma_{2n}|+2)\right) \\
&\leqq \log_2\left((|\gamma_1|+2)(|\gamma_2|+2)\ldots(|\gamma_{2n}|+2)\right) \\
&= \sum_{t=1}^{2n} \log_2(|\gamma_t|+2) \\
&\leqq \sum_{t=1}^{2n} \left(1+\log_2(|\gamma_t|+1)\right) \\
&\leqq L_1 \ .
\end{aligned}$$

So the total number of digits needed to specify the data in the LCP $(\bar{q}, \overline{M})$ in binary encoding is at most L_2.

From well known results the absolute value of the determinant of any square submatrix of B is at most $\frac{2^{\mathbf{L_1}}}{n}$. See Chapter 15 in [2.26]. So there exists a positive integer $\gamma < \frac{2^{\mathbf{L_1}}}{n}$ such that all the data in the system

$$\begin{aligned}
\gamma B^{-1}x &\geqq 0 \\
B^T(x-b) &\geqq 0
\end{aligned} \tag{8.2}$$

are integers. The absolute value of each entry in γB^{-1} is $< \left(\frac{2^{\mathbf{L_1}}}{n}\right)^{\mathbf{2}}$ (since it is less than or equal to a subdeterminant of B times γ). Hence the size of (8.2) the total number of digits in the data in it, in binary encoding, is at most L_3.

Theorem 8.1 **K** *has nonempty interior.*

Proof. Proving this theorem is equivalent to showing that there exists an $x \in \mathbf{R}^n$ satisfying each of the constraints in the definition of **K** as a strict inequality. This holds iff the system

$$\begin{aligned} B^{-1}x \qquad\qquad &> 0 \\ B^T x - B^T b x_{n+1} &> 0 \\ x_{n+1} &> 0 \end{aligned}$$

has a feasible solution $(x, x_{n+1}) = X$. By Motzkin's theorem of the alternatives (Theorem 5 of Appendix 1) this system has a feasible solution X iff there exists no row vectors $\pi, \mu \in \mathbf{R}^n$, $\delta \in \mathbf{R}^1$ satisfying

$$\begin{aligned} \pi B^{-1} + \mu B^T \qquad &= 0 \\ - \mu B^T b + \delta &= 0 \\ (\pi, \mu, \delta) \quad &\geqq 0 \end{aligned} \tag{8.3}$$

From the first set of constraints in this system we have $\mu B^T B = -\pi \leqq 0$. Since $B^T B$ is PD, we know that $\mu B^T B \leqq 0$, $\mu \geqq 0$ implies that μ, must be 0 in any feasible solution of (8.3). This in turn implies that π, δ will have to be zero too, a contradiction. So (8.3) has no feasible solution, hence **K** has a nonempty interior.

□

Theorem 8.2 $\mathbf{K} \cap \mathbf{E} = \mathbf{K} \cap \mathrm{Bd}(\mathbf{E}) = \{\bar{x}\}$.

Proof. By the results in Chapter 7, $(\overline{w}, \bar{z})$ is the solution of the LCP $(\bar{q}, \overline{M})$. So $\bar{z} = B^{-1}\bar{x} \geqq 0$, $0 \leqq \overline{w} = \bar{q} + \overline{M}\bar{z} = -B^T b + B^T B B^{-1}\bar{x} = B^T(\bar{x} - b)$. Also $(\bar{x} - \frac{b}{2})^T(\bar{x} - \frac{b}{2}) - (\frac{b^T b}{4}) = \bar{x}^T\bar{x} - \bar{x}^T b = \bar{x}^T(\bar{x} - b) = \bar{z}^T B^T(\bar{x} - b) = \bar{z}^T\overline{w} = 0$. So $\bar{x} \in \mathbf{K} \cap \mathbf{E}$.

Conversely, suppose $\hat{x} \in \mathbf{K} \cap \mathbf{E}$. Define $\hat{z} = B^{-1}\hat{x}$, $\widehat{w} = B^T(\hat{x} - b)$. Since $\hat{x} \in \mathbf{E}$ we have $0 \geqq (\hat{x} - \frac{b}{2})^T(\hat{x} - \frac{b}{2}) - (\frac{b^T b}{4}) = \hat{x}^T(\hat{x} - b) = \hat{z}^T\widehat{w}$. Since $\hat{x} \in \mathbf{K}$, we have $\hat{z} \geqq 0$, $\widehat{w} \geqq 0$, and hence $\hat{z}^T\widehat{w} \geqq 0$. These two together imply that $\hat{z}^T\widehat{w} = 0$ and we can verify that $\widehat{w} = B^T(\hat{x} - b) = \bar{q} + \overline{M}\hat{z}$. These facts together imply that $(\widehat{w}, \hat{z})$ is the solution of the LCP $(\bar{q}, \overline{M})$. Since M is PD, by Theorem 3, the LCP $(\bar{q}, \overline{M})$ has a unique solution and so $(\widehat{w}, \hat{z}) = (\overline{w}, \bar{z})$. So $\hat{x} = \bar{x}$. Thus $\mathbf{K} \cap \mathbf{E} = \{\bar{x}\}$. Also, for all $x \in \mathbf{K}$ we have $(x - \frac{b}{2})^T(x - \frac{b}{2}) = x^T(x - b) + (\frac{b^T b}{4}) = (B^{-1}x)^T B^T(x - b) + (\frac{b^T b}{4}) \geqq (\frac{b^T b}{4})$. This implies that $\mathbf{K} \cap \mathbf{E} = \mathbf{K} \cap \mathrm{Bd}(\mathbf{E})$.

□

Theorem 8.3 $\bar{x}$ *is an extreme point of* **K**.

Proof. Since M is PD, $(\overline{w}, \bar{z})$, the unique solution of the LCP $(\bar{q}, \overline{M})$ defined above, is a complementary BFS. So $\bar{z}$ is an extreme point of $\{z : -\overline{M}z \leqq \bar{q}, z \geqq 0\} = \mathbf{\Gamma}$. It can be verified that $z \in \mathbf{\Gamma}$ iff $x = Bz \in \mathbf{K}$. So there is a unique nonsingular linear transformation between $\mathbf{\Gamma}$ and $\mathbf{K}$. This, and the fact that $\bar{z}$ is an extreme point of $\mathbf{\Gamma}$ implies that $\bar{x} = B^{-1}\bar{z}$ is an extreme point of $\mathbf{K}$.

□

Theorem 8.4 *If $(\widetilde{w} = (\widetilde{w}_i), \tilde{z} = (\tilde{z}_i))$ is any extreme point of*

$$\begin{aligned} w - \overline{M}z &= \bar{q} \\ w \geqq 0, \quad z &\geqq 0 \end{aligned} \tag{8.4}$$

then $\widetilde{w}_i$, $\tilde{z}_i$, is either 0 or $> 2^{-\mathbf{L_2}}$, for each i.

Proof. As discussed above, L_2 is the size of the system (8.4). This result follows from the results discussed in Chapter 15 of [2.26].

□

Theorem 8.5 *The Euclidean lenght of any edge of $\mathbf{K}$ is $\geqq 2^{-\mathbf{L_3}}$.*

Proof. If the edge is unbounded, the theorem is trivially true. Each bounded edge of $\mathbf{K}$ is the line segment joining two distinct adjacent extreme points of $\mathbf{K}$. Let x^1, x^2 be two distinct adjacent extreme points of $\mathbf{K}$. Since $\mathbf{K}$ is the set of feasible solutions of (8.2), the results discussed in Chapter 15 of [2.26] imply that $x^1 = (\frac{u_{11}}{v_1}, \ldots, \frac{u_{n1}}{v_1})$, $x^2 = (\frac{u_{12}}{v_2}, \ldots, \frac{u_{n2}}{v_2})$ where all the u_{ij}'s are integers, v_1, v_2 are nonzero integers, all $|u_{ij}|$, $|v_1|$, $|v_2|$ are $\leqq \frac{2^{\mathbf{L_3}}}{n}$. Also, since $x^1 \neq x^2$, these facts imply that there exists a j satisfying $|x_j^1 - x_j^2| \geqq 2^{-\mathbf{L_3}}$. This clearly implies that $||x^1 - x^2|| \geqq 2^{-\mathbf{L_3}}$.

□

Theorem 8.6 *If $\varepsilon < 2^{-\mathbf{2(n+1)^2L_1}}$, the n-dimensional volume of $\mathbf{K} \cap \mathbf{E}_1 \geqq \varepsilon^{\mathbf{n}}$ $2^{-\mathbf{(n+1)L_3}}$.*

Proof. $\mathbf{K} \cap \mathrm{Bd}(\mathbf{E}) = \{\bar{x}\}$ and $\mathbf{K}$ has a nonempty interior. So $\mathbf{K} \cap \mathbf{E}_1$ contains all the points in $\mathbf{K}$ is an ε-neighbourhood of $\bar{x}$, and hence has a nonempty interior and a positive n-dimensional volume.

If one takes a sphere of radius α, a concentric sphere of radius $\alpha + \varepsilon$, and a hyperplane tangent to the smaller sphere at a boundary point x on it, then a tight upper bound on the distance between x and any point in the larger sphere on the side of the hyperplane opposite the smaller sphere is $\sqrt{2\alpha + \varepsilon^{\mathbf{2}}}$. Also the radius of $\mathbf{E}$ is $\sqrt{\frac{b^Tb}{4}} < 2^{(\mathbf{L_1-1})}$. $\bar{x}$ is an extreme point of $\mathbf{K}$, and every edge of $\mathbf{K}$ through $\bar{x}$, has a length $\geqq 2^{-\mathbf{L_3}}$ by Theorem 8.5. These facts and the choice of ε here, together imply that every edge of $\mathbf{K}$ through $\bar{x}$ intersects the boundary of $\mathbf{E}_1$. Let $V_1, \ldots, V_n$ be points along the edges of $\mathbf{K}$ through $\bar{x}$ that intersect the boundary of $\mathbf{E}_1$, at a distance of at most 1 but greater than ε from $\bar{x}$, such that $\{\bar{x}, V_1, \ldots, V_n\}$ is affinely independent. The portion of the edge between $\bar{x}$ and V_i lies inside $\mathbf{E}_1$ for at least a length of ε. See Figure 8.1. If $V_i(\varepsilon)$ is the point on the edge joining $\bar{x}$ and V_i at a distance of ε from $\bar{x}$, the volume of $\mathbf{E}_1 \cap \mathbf{K}$ is greater than or equal to the volume of the simplex whose vertices are $\bar{x}$, $V_i(\varepsilon)$ for $i = 1$ to n. From the choice of V_i, $V_i(\varepsilon) - \bar{x} = \gamma(V_i - \bar{x})$ where

$\gamma \geqq \varepsilon$. So in this case the volume of $\mathbf{E}_1 \cap \mathbf{K}$ is greater than or equal to

$$\begin{aligned} & \frac{1}{n!}\left|\text{determinant of } \Big(\varepsilon(V_1 - \bar{x}) \;\vdots\; \ldots \;\vdots\; \varepsilon(V_n - \bar{x})\Big)\right| \\ = \;& \frac{\varepsilon^{\mathbf{n}}}{n!} \left| \text{ determinant of } ((V_1 - \bar{x}) \;\vdots\; \ldots \;\vdots\; (V_n - \bar{x}))\right| \\ = \;& \frac{\varepsilon^{\mathbf{n}}}{n!}\left| \text{ determinant of } \begin{pmatrix} 1 & 1 & \vdots & \ldots & \vdots & 1 \\ \bar{x} & V_1 & \vdots & \ldots & \vdots & V_n \end{pmatrix} \right| \\ > \;& \varepsilon^{\mathbf{n}} 2^{-(\mathbf{n}+\mathbf{1})\mathbf{L_3}} \, . \end{aligned}$$

using the results from Chapter 15 in [2.26].

□

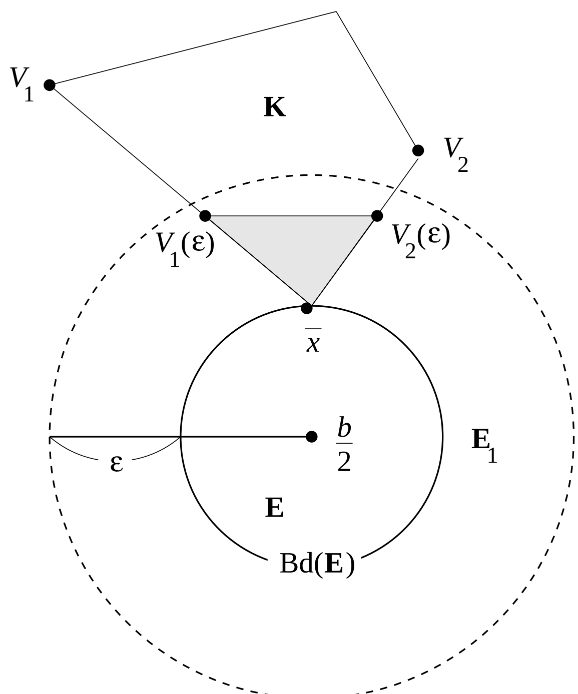

Figure 8.1 The volume of $\mathbf{E}_1 \cap \mathbf{K}$ is greater than or equal to the volume of the shaded simplex.

Theorem 8.7 *Let $\hat{x} \in \mathbf{E}_1 \cap \mathbf{K}$, $\hat{z} = B^{-1}\hat{x}$, $\widehat{w} = B^T(\hat{x} - b)$. Then, for all $j = 1$ to n*

$$\begin{aligned} |\hat{x}_j - \bar{x}_j| &\leqq 2^{\mathbf{L_1}}\sqrt{\varepsilon} \\ |\hat{z}_j - \bar{z}_j| &\leqq n2^{\mathbf{2L_1}}\sqrt{\varepsilon} \\ |\widehat{w}_j - \overline{w}_j| &\leqq n2^{\mathbf{2L_1}}\sqrt{\varepsilon} \end{aligned}$$

Proof. As mentioned earlier, the absolute value of any entry in B^{-1} is $\leqq 2^{\mathbf{L_1}}$, and the same fact obviously holds for B^T. The radius of $\mathbf{E}$ is $\frac{b^Tb}{4} < 2^{\mathbf{L_1 - 1}}$. The results in this theorem follow from these facts and the definitions of $\mathbf{E}$, $\mathbf{E}_1$, $\widehat{w}$, $\hat{z}$.

□

Theorem 8.8 *Let* $\hat{x} \in \mathbf{E}_1 \cap \mathbf{K}$ *and* $\hat{z} = B^{-1}\hat{x}$. *If* $\varepsilon \leqq 2^{-\mathbf{2(n+1)^2(L_1+1)}}$, *then*

$$\begin{aligned} \hat{z}_j &\leqq (\tfrac{1}{4})2^{-\mathbf{L_2}}, & &\text{for } j \text{ such that } \bar{z}_j = 0 \\ \hat{z}_j &\geqq (\tfrac{3}{4})2^{-\mathbf{L_2}} = \delta, & &\text{for } j \text{ such that } \bar{z}_j > 0\ . \end{aligned}$$

Proof. This follows from Theorems 8.7 and 8.4.

□

The Algorithm

Fix $\varepsilon = 2^{-\mathbf{2(n+1)^2(L_1+1)}}$. Consider the following system of constraints.

$$-B^{-1}x \leqq 0, \qquad B^T(x-b) \leqq 0 \tag{8.5}$$

$$(x-\frac{b}{2})^T(x-\frac{b}{2}) \leqq \left(\varepsilon + \sqrt{\frac{b^Tb}{4}}\right)^{\mathbf{2}} \tag{8.6}$$

Any point $\hat{x} \in \mathbf{R}^n$ satisfying both (8.5) and (8.6) is in $\mathbf{K} \cap \mathbf{E}_1$. We use an ellipsoid method to first find such a point $\hat{x}$. Then using $\hat{x}$ we compute $\bar{x}$ in a final step.

Define $x^1 = \frac{b}{2}$, $A_1 = I\left(\varepsilon + \sqrt{\frac{b^Tb}{4}}\right)^{\mathbf{2}}$, where I is the unit matrix of order n, $N = 8(n+1)^{\mathbf{4}}(L_1+1)$. Go to Step 2.

General Step $r+1$

Let x^r, A^r, $\mathbf{E}_r = \mathbf{E}(x^r, A_r)$ be respectively the center, positive definite symmetric matrix, and the ellipsoid at the beginning of this step. If x^r satisfies both (8.5), (8.6), terminate the ellipsoid method, call x^r as $\hat{x}$ and with it go to the final step described below. If x^r violates (8.5) select a constraint in it that it violates most, breaking ties arbitrarily, and suppose it is $ax \leqq d$. If x^r satisfies (8.5) but violates (8.6), find the point of intersection ξ^r, of the line segment joining x^1 and x^r with the boundary of $\mathbf{E}_1$. So $\xi^r = \lambda x^1 + (1-\lambda)x^r$ where $\lambda = 1 - \frac{\varepsilon + \sqrt{\frac{b^Tb}{4}}}{||x^r - x^1||}$. Find the tangent plane of $\mathbf{E}_1$ at its boundary point ξ^r, and find out the half-space determined by this hyperplane which does not contain the point x^r. Suppose this half-space is determined by the constraint "$ax \leqq d$". See Figure 8.2.

Now define

$$\begin{aligned} \gamma_r &= \frac{d - ax^r}{\sqrt{aA_ra^T}} \\ x^{r+1} &= x^r - \left(\frac{1-\gamma_r n}{1+n}\right)\frac{A_ra^T}{\sqrt{aA_ra^T}} \\ A_{r+1} &= \frac{(1-\gamma_r^2)n^2}{n^2-1}\left(A_r - \left(\frac{2}{n+1}\right)\left(\frac{1-\eta\gamma_r}{1-\gamma_r}\right)\frac{(A_ra^T)(A_ra^T)^T}{aA_ra^T}\right) \end{aligned} \tag{8.7}$$

where the square root of a quantity always represents the positive square root of that quantity. With x^{r+1}, A_{r+1}, $\mathbf{E}_{r+1} = \mathbf{E}(x^{r+1}, A_{r+1})$ move to the next step in the ellipsoid method.

After at most N steps, this ellipsoid method will terminate with the point x^r in the terminal step lying in $\mathbf{E}_1 \cap \mathbf{K}$. Then go to the final step discussed below.

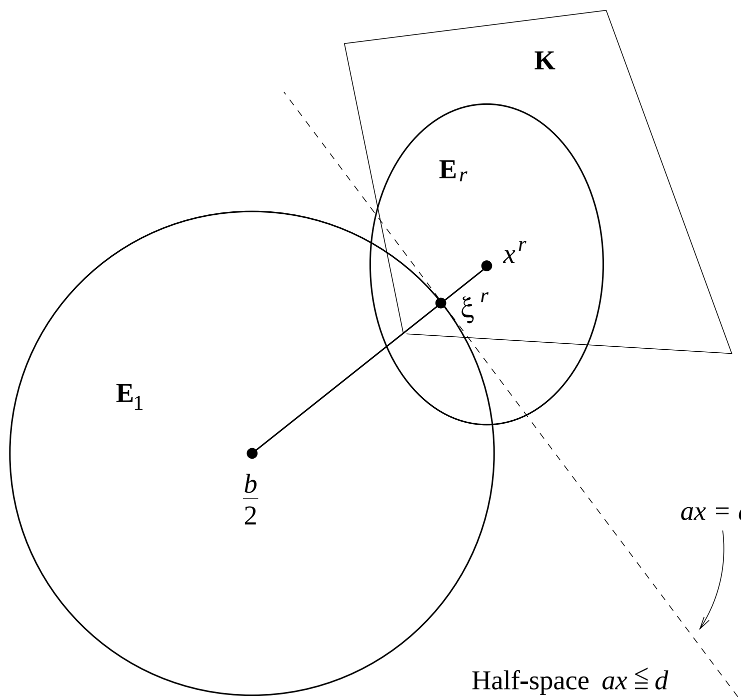

Figure 8.2 Construction of "$ax \leqq d$" when x^r satisfies (8.5) but violates (8.6).

Final Step : Let the center of the ellipsoid in the terminal step be $\hat{x}$ (this is the point x^r in the last step r of the ellipsoid method). Let $\hat{z} = B^{-1}\hat{x}$. Let $\mathbf{J} = \{j : j$ such that $\hat{z}_j \geqq \delta\}$. Let $y_j = z_j$ if $j \in \mathbf{J}$, w_j if $j \notin \mathbf{J}$ and let $y = (y_1, \dots, y_n)$. Then y is a complementary feasible basic vector for the LCP $(\bar{q}, \overline{M})$, and the BFS of (8.4) corresponding to y is the solution of this LCP. If this solution is $(\overline{w}, \bar{z})$, $\bar{x} = B\bar{z}$ is the nearest point in Pos(B) to b.

Definition We denote by e, the base of natural logarithms. $\mathrm{e} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}$, it is approximately equal to 2.7.

Proof of the Algorithm

Let x^r, A_r, $\mathbf{E}_r = \mathbf{E}(x^r, A_r)$, be the center, positive definite symmetric matrix, and the ellipsoid at the beginning of step $r+1$. The inequality "$ax \leqq d$" is choosen in this step $r+1$ in such a way that x^r violates it. In the hyperplane "$ax = d$" decrease d until a value d_1 is reached such that the translate "$ax = d_1$" is a tangent plane to the ellipsoid $\mathbf{E}_r$, and suppose the boundary point of $\mathbf{E}_r$ where this is a tangent plane is η_r. Then $\mathbf{E}_{r+1} = \mathbf{E}(x^{r+1}, A_{r+1})$ is the minimum volume ellipsoid that contains $\mathbf{E}_r \cap \{x : ax \leqq d\}$, the shaded region in Figure 8.3, it has η_r as a boundary point and has the same tangent plane at η_r as $\mathbf{E}_r$. From the manner in which the inequality "$ax \leqq d$" is selected, it is clear that if $\mathbf{E}_r \supset \mathbf{E}_1 \cap \mathbf{K}$, then $\mathbf{E}_{r+1} \supset \mathbf{E}_1 \cap \mathbf{K}$. Arguing inductively on r, we conclude that every ellipsoid $\mathbf{E}_r$ constructed during the algorithm satisfies $\mathbf{E}_r \supset \mathbf{E}_1 \cap \mathbf{K}$. From Theorem 8.6, the volume of $\mathbf{E}_1 \cap \mathbf{K}$ is $\geqq 2^{-\mathbf{4n(n+1)^2(L_1+1)}}$. From the results in Chapter 15 of [2.26] we know that the volume of $\mathbf{E}_r$ gets multiplied by a factor of $e^{-\frac{\mathbf{1}}{\mathbf{2(n+1)}}}$ or less, after each step in the ellipsoid method. $\mathbf{E}_1$ is a ball whose radius is $(\varepsilon + \sqrt{\frac{b^Tb}{4}})$, and $b^Tb < 2^{\mathbf{2L_1}}$. So the volume of $\mathbf{E}_1$ is at most $2^{\mathbf{2nL_1}}$. The algorithm terminates in step r, if the center x^r satisfies (8.5), (8.6) and that is, it is a point in $\mathbf{E}_1 \cap \mathbf{K}$. If termination does not occur up to step $N = 8(n+1)^{\mathbf{4}}(L_1+1)$, the volume of $\mathbf{E}_N$ is at most $2^{\mathbf{2L_1n}}e^{-\frac{\mathbf{N}}{\mathbf{2(n+1)}}} < 2^{-\mathbf{4n(n+1)^2(L_1+1)}}$. From the fact that the volume of $\mathbf{E}_1 \cap \mathbf{K} > 2^{-\mathbf{4n(n+1)^2(L_1+1)}}$ this is a contradiction to $\mathbf{E}_N \supset \mathbf{E}_1 \cap \mathbf{K}$. So for some $r \leqq N$, we will have $x^r \in \mathbf{E}_1 \cap \mathbf{K}$, and in that step the ellipsoid method terminates. The validity of the remaining portion of the algorithm follows from Theorem 8.7, 8.8, 2.9. Since the ellipsoid method terminates after at most $N = 8(n+1)^{\mathbf{4}}(L_1+1)$ steps, the algorithm is obviously polynomially bounded.

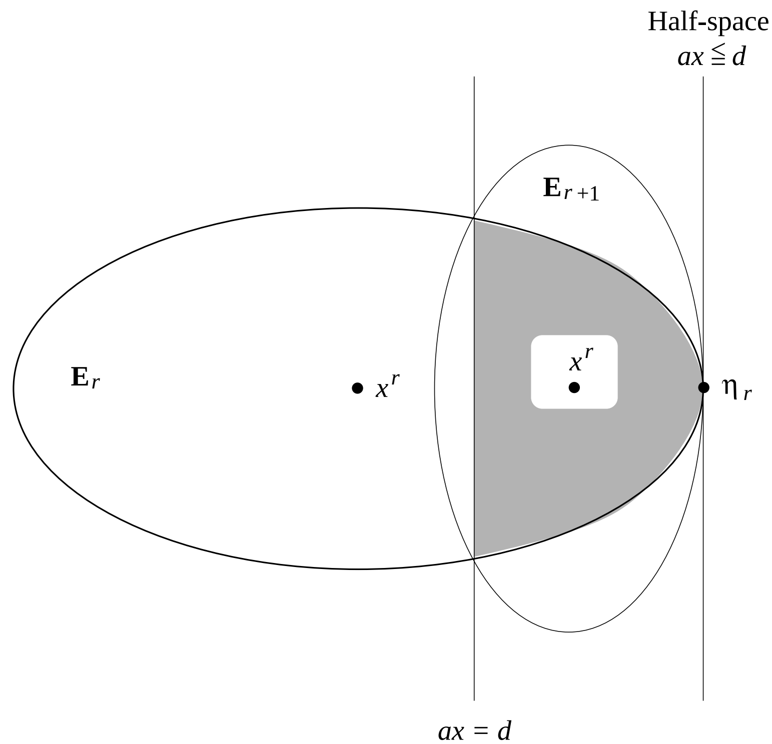

Figure 8.3 Construction of the new ellipsoid $\mathbf{E}_{r+1}$

In practice, it is impossible to run the algorithm using exact arithmetic. To run the algorithm using finite precision arithmetic, all computations have to be carried out to a certain number of significant digits as discussed in [8.13], and the ellipsoid have to be expanded by a small amout in each iteration (this is achieved by multiplying the matrix A_r in each step by a number slightly larger than one in each step). As pointed out in [2.26] if each quantity is computed correct to $61nL_1$ bits of precision, and D_{r+1} multiplied by $(1 + \frac{1}{16n^2})$ before being rounded, all the results continue to hold.

Computational Comparison

Y. Fathi [8.10] did a comparative study in which this ellipsoid algorithm has been compared with the algorithm discussed in Chapter 7 for the nearest point problem. We provide a summary of his results here. In the study the matrix B was generated randomly, with its entries to be integers between -5 and $+5$. The b-vector was also generated randomly with its entries to be integers between -20 and $+20$. Instead of using computer times for the comparison, he counted the number of iterations of various types and from it estimated the total number of multiplication and division operations required before termination on each problem. Problems with $n = 10$, 20, 30, 40, 50 were tried and each entry in the table is an average for 50 problems. Double precision was used. It was not possible to take the values of ε and δ as small as those recomended in the algorithm. Mostly he tried $\varepsilon, \delta = 0.1$ (the computational effort before termination in the ellipsoid algorithms reported in the table below refers to $\varepsilon, \delta = 0.1$), and with this, sometimes the complementary basic vector obtained at termination of the algorithm turned out to be infeasible (this result is called an **unsuccessful run**). He noticed that if the values of these tolerances were decreased, the probability of an unsuccessful run decreases; but the computational effort required before termination increases very rapidly.

	Average Number of Multiplication and Division Operations Required Before Termination in	
n	The Algorithm of Chapter 7	The Ellipsoid Algorithm
10	Too small	33,303
20	16,266	381,060
30	42,592	1,764,092
40	170,643	5,207,180
50	324,126	11,286,717

These empirical results suggest that the ellipsoid algorithm cannot compete with the algorithm discussed in Chapter 7 for the nearest problem, in practical efficiency. The same comment seems to hold for the other ellipsoid algorithms discussed in the following sections.

8.5 An Ellipsoid Algorithm for LCPs Associated with PD Matrices

In this section $M = (m_{ij})$ denotes a given PD matrix of order n (symmetric or not) with integer entries, and $q = (q_i)$ denotes a given nonzero integer column vector in $\mathbf{R}^n$. We consider the LCP (q, M).

Definitions

Let ε be a small positive number. Later on we specify how small ε should be. Let

$$\mathbf{K} = \{z : \; Mz + q \geqq 0, \; z \geqq 0\}.$$
$$(\overline{w} = M\bar{z} + q, \bar{z}) = \text{unique solution of the LCP } (q, M).$$
$$f(z) = z^T(Mz + q).$$
$$\mathbf{E} = \{z : \; f(z) \leqq 0\}.$$
$$\text{Bd}(\mathbf{E}) = \text{Boundary of } \mathbf{E} = \{z : \; f(z) = 0\}.$$
$$L = \left\lceil (1 + \log_2 n) + \sum_{i,j}\big(1 + \log_2(|m_{ij}| + 1)\big) + \sum_{i}\big(1 + \log_2(|q_i| + 1)\big) \right\rceil$$
$$\mathbf{E}_\varepsilon = \{z : \; z^T(Mz + q) \leqq \varepsilon\} \text{ for } \varepsilon > 0.$$
$$\mathbf{E}_0 = \{z : \; z^T z \leqq 2^{\mathbf{2L}}\}.$$

Since M is a PD matrix, $\mathbf{E}$ defined above is an ellipsoid.

Some Preliminary Results

Theorem 8.9 *The set* $\mathbf{K} = \{z : Mz + q \geqq 0, z \geqq 0\}$ *has nonempty interior.*

Proof. Remembering that M is a PD matrix, the proof of this theorem is similar to the proof of Theorem 8.1 of Section 8.4.

□

Theorem 8.10 $\mathbf{E} \cap \mathbf{K} = \text{Bd}(\mathbf{E}) \cap \mathbf{K} = \{\bar{z}\}$.

Proof. This follows directly from the definitions.

□

Theorem 8.11 *$\bar{z}$ is an extreme point of* $\mathbf{K}$. *Also, every extreme point z of* $\mathbf{K}$ *other than $\bar{z}$ satisfies* $f(z) > 2^{-\mathbf{2L}}$.

Proof. Since $(\overline{w}, \bar{z})$ is a BFS of: $w - Mz = q$, $w \geqq 0$, $z \geqq 0$; $\bar{z}$ is an extreme point of $\mathbf{K}$. Also, L is the size of this system. Since $(\overline{w}, \bar{z})$ is the unique solution of the LCP (q, M), at every extreme point z of $\mathbf{K}$ other than $\bar{z}$, we must have $f(z) > 0$. Using arguments similar to these in Theorem 8.4 of Section 8.4, we conclude that for each i, either z_i is 0 or $> 2^{-\mathbf{L}}$, and $M_{i.}z + q_i$ is 0 or $> 2^{-\mathbf{L}}$, at every extreme point z of $\mathbf{K}$. Combining these results we conclude that every extreme point z of $\mathbf{K}$ other than $\bar{z}$ satisfies $f(z) > 2^{-\mathbf{2L}}$.

□

Theorem 8.12 *For $0 < \varepsilon \leqq 2^{-\mathbf{2L}}$, the n-dimensional volume of $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K}$ is $\geqq \varepsilon^{\mathbf{n}} 2^{-\mathbf{3(n+1)L}}$.*

Proof. Obviously $\bar{z} \in \mathbf{E}_\varepsilon \cap \mathbf{K}$, and by Theorem 8.11, no other extreme point z of $\mathbf{K}$ lies in $\mathbf{E}_\varepsilon \cap \mathbf{K}$ for $0 < \varepsilon \leqq 2^{-\mathbf{2L}}$. So for every value of ε in the specified range, every edge of $\mathbf{K}$ through $\bar{z}$ intersects $\mathbf{E}_\varepsilon$. Also, since $\mathbf{K}$ has a nonempty interior by Theorem 8.9, $\mathbf{E}_\varepsilon \cap \mathbf{K}$ has a positive n-dimensional volume, $\mathbf{K}$ might be unbounded, but by the results in Chapter 15 of [2.26], at every extreme point of $\mathbf{K}$, both z_i and $M_{i.} z + q_i$ are $\leqq \frac{2^{\mathbf{L}}}{n}$ for each i. Let $\widehat{\mathbf{K}} = \{z : 0 \leqq z_j \leqq \frac{2^{\mathbf{L}}}{n}, 0 \leqq M_{j.} z + q_j \leqq \frac{2^{\mathbf{L}}}{n}, \text{ for } j = 1 \text{ to } n\}$. By the above facts, every edge of $\widehat{\mathbf{K}}$ through z is either an edge $\mathbf{K}$ (if it is a bounded edge of $\mathbf{K}$), or a portion of an edge of $\mathbf{K}$ (if it is an unbounded edge of $\mathbf{K}$). Let $z^1, \ldots, z^n$ be adjacent extreme points of $\bar{z}$ in $\widehat{\mathbf{K}}$, such that $\{\bar{z} : z^1, \ldots, z^n\}$ is affinely independent. The above facts imply that all these points $\bar{z}$, z^t, $t = 1$ to n are in $\mathbf{E}_0$. Since M is PD, $f(z)$ is convex. Let $\lambda = \varepsilon 2^{-\mathbf{2L}}$. So for each $t = 1$ to n, $f(\bar{z} + \lambda(z^t - \bar{z})) \leqq (1-\lambda) f(\bar{z}) + \lambda f(z^t) = \lambda f(z^t) = \lambda \sum_{i=1}^n z_i^t (M_{i.} z^t + q_i) \leqq \lambda \sum_{i=1}^n \left(\frac{2^{\mathbf{L}}}{n}\right)\left(\frac{2^{\mathbf{L}}}{n}\right) \leqq \varepsilon$. This implies that the line segment $[\bar{z}, \bar{z} + \lambda(z^t - \bar{z})]$ completely lies inside $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K}$. So the volume of $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K} \geqq$ the volume of the simplex whose vertices are $\bar{z}$, $\bar{z} + \lambda(z^t - \bar{z})$, $t = 1$ to n, which is

$$\begin{aligned}
&= \frac{1}{n!} \left| \text{ determinant of } \left(\lambda(z^1 - \bar{z}) \quad \vdots \quad \ldots \quad \vdots \quad \lambda(z^t - \bar{z})\right) \right| \\
&\geqq \lambda^{\mathbf{n}} 2^{-\mathbf{(n+1)L}}, \text{ by results similar to those in the proof of Theorem 8.6} \\
&\geqq \varepsilon^{\mathbf{n}} 2^{-\mathbf{(3n+1)L}} .
\end{aligned}$$

□

Theorem 8.13 *Let $\varepsilon_0 = 2^{-\mathbf{(6L+1)}}$. For any point $\hat{z} \in \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$, we have:*

$$\begin{array}{rrcl}
\textit{either} & \hat{z}_i \leqq \sqrt{\varepsilon_0} & < & 2^{-\mathbf{3L}} \\
\textit{or} & M_{i.} \hat{z} + q_i \leqq \sqrt{\varepsilon_0} & < & 2^{-\mathbf{3L}} .
\end{array}$$

Proof. For any i, if both $\hat{z}_i$ and $M_{i.} \hat{z} + q_i$ are $> \sqrt{\varepsilon_0}$, then $\hat{z}(M\hat{z} + q) > \varepsilon_0$, contradiction to the fact that $\hat{z} \in \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$.

□

Theorem 8.14 *Let $\hat{z}$ by any point in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$. Define*

$$y_i = \begin{cases} w_i & \text{if } \hat{z}_i < 2^{-\mathbf{3L}} \\ z_i & \text{if } \hat{z}_i \geqq 2^{-\mathbf{3L}} . \end{cases}$$

Then $(y_1, \ldots, y_n)$ is a complementary feasible basic vector for the LCP (q, M).

Proof. Let $\mathbf{J}_1 = \{i : \hat{z}_i \geqq 2^{-\mathbf{3L}}\}$, $\mathbf{J}_2 = \{i : \hat{z}_i < 2^{-\mathbf{3L}}\}$. So $\mathbf{J}_1 \cap \mathbf{J}_2 = \emptyset$ and $\mathbf{J}_1 \cup \mathbf{J}_2 = \{1, \ldots, n\}$, and by Theorem 8.13, $M_{i.} \hat{z} + q_i < 2^{-\mathbf{3L}}$ for $i \in \mathbf{J}_1$.

In [8.11] P. Gács and L. Lovász proved the following lemma :

Consider the system of constraints

$$A_{i.}x \leqq b_i, \quad i = 1 \text{ to } m \tag{8.8}$$

with integer data, and let l be the size of this system. Suppose $\hat{x}$ is a solution of

$$A_{i.}x \leqq b_i + 2^{-\mathbf{l}}, \quad i = 1 \text{ to } m$$

such that $A_{i.}x \geqq b_i$, $i = 1$ to k, and let $\{A_{p_1.}, \dots, A_{p_r.}\} \subset \{A_{1.}, \dots, A_{k.}\}$ be such that it is linearly independent and it spans $\{A_{1.}, \dots, A_{m.}\}$ linearly. Let $\bar{x}$ be any solution of the system of equations

$$A_{p_t.}x = b_{p_t}, \quad t = 1 \text{ to } r \ .$$

Then $\bar{x}$ is a solution (8.8). See also Chapter 15 in [2.26]. We will use this lemma in proving this theorem. Consider the system :

$$\begin{aligned} -M_{i.}z &\leqq q_i + 2^{-\mathbf{3L}}, \quad i = 1 \text{ to } n \\ -z_i &\leqq 0 + 2^{-\mathbf{3L}}, \quad i = 1 \text{ to } n \\ M_{i.}z &\leqq -q_i + 2^{-\mathbf{3L}}, \quad i \in \mathbf{J}_1 \\ z_i &\leqq 0 + 2^{-\mathbf{3L}}, \quad i \in \mathbf{J}_2 \ . \end{aligned} \tag{8.9}$$

We know that $\hat{z}$ solves this system and in addition $\hat{z}$ also satisfies $M_{i.}\hat{z} \geqq -q_i$, $i \in \mathbf{J}_1$ and $\hat{z} \geqq 0$, $i \in \mathbf{J}_2$. Also, since M is PD, the set $\{M_{i.} : i \in \mathbf{J}_1\} \cup \{I_{.i} : i \in \mathbf{J}_2\}$ is linearly independent and linearly spans all the row vectors of the constraint coefficient matrix of the system (8.9). From the lemma of P. Gács and L. Lovász mentioned above, these facts imply that if $\tilde{z}$ is a solution of the system of equations :

$$\begin{aligned} M_{i.}z &= -q_i, \ i \in \mathbf{J}_1 \\ z_i &= 0, \quad i \in \mathbf{J}_2 \end{aligned} \tag{8.10}$$

then $\tilde{z}$ also satisfies :

$$\begin{aligned} -M_{i.}z &\leqq q_i, \ i = 1 \text{ to } n \\ -z_i &\leqq 0, \ i = 1 \text{ to } n \end{aligned}$$

So $\tilde{z} \geqq 0$, $\widetilde{w} = M\tilde{z} + q \geqq 0$ and since $\tilde{z}_i = 0$ for $i \in \mathbf{J}_2$ and $M_{i.}z + q_i = 0$ for $i \in \mathbf{J}_1$ we have $f(\tilde{z}) = 0$ (since $\mathbf{J}_1 \cap \mathbf{J}_2 = \emptyset$ and $\mathbf{J}_1 \cup \mathbf{J}_2 = \{1, \dots, n\}$). So $(\widetilde{w}, \tilde{z})$ is the solution of the LCP (q, M). Since $\tilde{z}$ is the solution of (8.10), $(\widetilde{w}, \tilde{z})$ is the BFS of the system: $w - Mz = q$, $w \geqq 0$; $z \geqq 0$; corresponding to the basic vector y. So y is a complementary feasible basic vector for the LCP (q, M).

□

The Algorithm

Fix $\varepsilon = \varepsilon_0 = 2^{-\mathbf{(6L+1)}}$. So $\mathbf{E}_0 = \mathbf{E}(0, 2^{\mathbf{2L}} I)$. Define $N = 2(n+1)^{\mathbf{2}}(11L+1)$ in this section. With $z^0 = 0$, $A_0 = 2^{\mathbf{2L}} I$, $\mathbf{E}(z^0, A_0)$ go to Step 1.

General Step $r+1$ **:** Let z^r, A_r, $\mathbf{E}_r = \mathbf{E}(z^r, A_r)$; be respectively the center, PD symmetric matrix, and the ellipsoid at the beginning of this step. If z^r satisfies :

$$\begin{aligned} -Mz - q &\leqq 0 \\ -q &\leqq 0 \end{aligned} \tag{8.11}$$

$$z^T(Mz+q) \leqq \varepsilon \tag{8.12}$$

terminate the ellipsoid algorithm, call z^r as $\hat{z}$ and go to the **final step** described below. If z^r violates (8.11), select a constraint in it that it violates most, breaking ties arbitrarily, and suppose it is "$az \leqq d$". If z^r satisfies (8.11) but violates (8.12), let ξ^r be the point of intersection of the line segment joining the center of the ellipsoid $\mathbf{E}_{\varepsilon_0}$ (this is, $z' = -\big(\frac{M+M^T}{2}\big)^{-1}(\frac{q}{2})$) and z^r with the boundary $\mathbf{E}_{\varepsilon_0}$. Therefore $\xi^r = \lambda z' + (1-\lambda)z^r$, where λ is the positive root of the equation $(\lambda z' + (1-\lambda)z^r)^T M(\lambda z' + (1-\lambda)z^r) + q = \varepsilon_0$. Let $az = d$ by the equation of the tangent hyperplane to $\mathbf{E}_{\varepsilon_0}$ at ξ^r, where the equation is written such that the half-space $az \leqq d$ does not contain z^r. Define γ_{r+1}, A_{r+1}, as in (8.7) and

$$z^{r+1} = z^r - \Big(\frac{1-\gamma_r n}{1+n}\Big)\Big(\frac{A_r a^T}{\sqrt{aA_r a^T}}\Big)$$

With z^{r+1}, A_{r+1}, $\mathbf{E}_{r+1} = \mathbf{E}(z^{r+1}, A_{r+1})$, move to the next step in the ellipsoid algorithm.

After at most N steps, this ellipsoid algorithm will terminate with the point z^r in the terminal step lying in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$. Then go to the final step described below.

Final Step: Let the center of the ellipsoid in the terminal step by $\hat{z}$. Using $\hat{z}$, find the complementary BFS as outlined in Theorem 8.14.

Proof of the Algorithm

The updating formulas used in this ellipsoid algorithm are the same as those used in the algorithm of Section 8.4. Hence using the same arguments as in Section 8.4, we can verify that $\mathbf{E}_r \supset \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$ for all r. The volume of $\mathbf{E}_0$ is $< 2^{\mathbf{2Ln}}$. After each step in the ellipsoid algorithm, the volume of the current ellipsoid $\mathbf{E}_r$ gets multiplied by a factor of $\mathrm{e}^{-\frac{\mathbf{1}}{\mathbf{2(n+1)}}}$ or less. So if the ellipsoid algorithm does not terminate even after N steps, the volume of $\mathbf{E}_N \leqq \mathrm{e}^{-\mathbf{(n+1)(11L+1)}} 2^{\mathbf{2Ln}} < 2^{-\mathbf{L(9n+1)-n}}$, contradiction to the fact that $\mathbf{E}_N \supset \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$ and Theorem 8.12. So for some $r \leqq N$, we will have $z^r \in \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}$, and in that step the ellipsoid algorithm terminates. Hence the algorithm is obviously polynomially bounded.

Comments made in Section 8.4 about the precision of computation required, remain valid here also.

8.6 An Ellipsoid Algorithm for LCPs Associated with PSD Matrices

In this section we consider the LCP (q, M) where M denotes a given PSD matrix of order n (symmetric or not) with integer entries, and q denotes a given integer column vector in $\mathbf{R}^n$.

Definitions

Let $\mathbf{K}$, $\mathbf{E}$, $\mathrm{Bd}(\mathbf{E})$, L, $\mathbf{E}_\varepsilon$ be as defined in Section 8.5. Let $\mathbf{E}_0 = \{z : z^T z \leqq 2^{\mathbf{2(L+1)}}\}$. Since M is only PSD here, $\mathbf{K}$ may have no interior, in fact $\mathbf{K}$ may even be empty. Also $\mathbf{E}$, $\mathbf{E}_\varepsilon$ may not be ellispoids. Let $e_n = (1, \ldots, 1)^T \in \mathbf{R}^n$.

Some Preliminary Results

Theorem 8.15 *In this case the LCP (q, M) has a solution iff $\mathbf{K} \neq \emptyset$. If $\mathbf{K} \neq \emptyset$, there exists a solution, $(\overline{w}, \bar{z})$, to the LCP (q, M) where $\bar{z}$ is an extreme point of $\mathbf{K}$. When $\mathbf{K} \neq \emptyset$, the LCP (q, M) may have many solutions, but the set of all solutions is a convex set which is $\mathbf{E} \cap \mathbf{K} = \mathrm{Bd}(\mathbf{E}) \cap \mathbf{K}$.*

Proof. Since M is PSD, the fact that (q, M) has a solution iff $\mathbf{K} \neq \emptyset$ follows from Theorem 2.1. When $\mathbf{K} \neq \emptyset$, the complementary pivot algorithm produces a solution $(\overline{w}, \bar{z})$, to the LCP (q, M) which is a BFS and this implies that $\bar{z}$ is an extreme point of $\mathbf{K}$. The set of all solutions of the LCP (q, M) is obviously $\mathrm{Bd}(\mathbf{E}) \cap \mathbf{K}$, and from the definition of $\mathbf{K}$, and $\mathbf{E}$ here it is clear that in this case $\mathrm{Bd}(\mathbf{E}) \cap \mathbf{K} = \mathbf{E} \cap \mathbf{K}$, and since both $\mathbf{E}$ and $\mathbf{K}$ are convex sets ($\mathbf{E}$ is convex because M is PSD), this set is convex.

□

Theorem 8.16 *When $\mathbf{K} \neq \emptyset$, $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K}$ contains all the extreme points z of $\mathbf{K}$ such that $(w = Mz + q, z)$ is a solution of the LCP (q, M).*

Proof. By the results discussed in Chapter 15 of [2.26] if $(\overline{w}, \bar{z})$ is solution of (q, M) which is BFS, then $z \in \mathbf{E}_0$. The rest follows from Theorem 8.15.

□

In this case $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K}$ may not contain all the z which lead to solutions of the LCP (q, M), Theorem 8.16 only guarantees that $\mathbf{E}_0 \cap \mathbf{E}_\varepsilon \cap \mathbf{K}$ contains all the z which are extreme points of $\mathbf{K}$ that lead to solutions of (q, M). Since M is PSD, the set of solutions of the LCP (q, M) may in fact be unbounded and hence all of it may not lie in $\mathbf{E}_0$.

Theorem 8.17 *If z_i is positive in some solution of (q, M), then its complement w_i is zero in all solutions of (q, M). Similarly if w_i is positive in some solutions of (q, M), then z_i is zero in all solutions of (q, M).*

Proof. By Theorem 8.15, the set of all solutions of (q, M) is convex set. So if (w^1, z^1), (w^2, z^2) are two solutions of (q, M) satisfying the properties that $z_i^1 > 0$ and $w_i^2 > 0$, then the other points on the line segment joining (w^1, z^1), (w^2, z^2) cannot be solutions of (q, M) (because they violate the complementarity constraint $w_i z_i = 0$) contradicting the fact that the set of solutions of (q, M) is a convex set.

□

Theorem 8.18 *If $\tilde{z}$ is an extreme point of $\mathbf{K}$, for each i either $\tilde{z}_i = 0$ or $2^{-\mathbf{L}} \leqq \tilde{z}_i \leqq \frac{2^{\mathbf{L}}}{n}$. Also either $M_{i.}\tilde{z} + q_i$ is zero or $2^{-\mathbf{L}} \leqq M_{i.}\tilde{z} + q_i \leqq \frac{2^{\mathbf{L}}}{n}$. Also at every extreme point $\tilde{z}$ of $\mathbf{K}$ that does not lead to a solution of (q, M), we will have $f(\tilde{z}) = \tilde{z}^T(Mz + q) > 2^{-\mathbf{2L}}$.*

Proof. Similar to the proof of Theorem 8.11 in Section 8.5.

□

Theorem 8.19 $\mathbf{K} \neq \emptyset$ *iff the set of solutions of*

$$\begin{aligned} Mz + q &\geqq -2^{-\mathbf{10L}} e \\ z &\geqq -2^{-\mathbf{10L}} e \end{aligned} \tag{8.13}$$

has a nonempty interior.

Proof. By the results of P. Gács and L. Lovász in [8.11] (also see Chapter 15 in [2.26]), (8.13) is feasible iff $\mathbf{K} \neq \emptyset$. Also any point in $\mathbf{K}$ is an interior point of the set of feasible solutions of (8.13).

□

Let $\mathbf{K}_1$ denote the set of feasible solutions of (8.13).

Theorem 8.20 *Let $\varepsilon_0 = 2^{-(\mathbf{6L+1})}$. For any point $\hat{z} \in \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$, we have for each $i = 1$ to n, either $\hat{z}_i < 2^{-\mathbf{3L}}$, or $M_{i.}\hat{z} + q_i < 2^{-\mathbf{3L}}$.*

Proof. Suppose that $\hat{z}_i \geqq 2^{-\mathbf{3L}}$ and $M_{i.}\hat{z} + q_i \geqq 2^{-\mathbf{3L}}$. Since $\hat{z} \in \mathbf{E}_{\varepsilon_0}$, $\hat{z}^T(M\hat{z} + q) \leqq 2^{-(\mathbf{6L+1})}$. Then we have $\sum_{t=1, t\neq i}^{n} \hat{z}_t(M_{t.}\hat{z} + q_t) \leqq 2^{-(\mathbf{6L+1})} - 2^{-\mathbf{6L}} \leqq -2^{-(\mathbf{6L+1})}$. But from (8.13) and the definition of $\mathbf{E}_0$ we arrive at the contradiction $\sum_{t=1, t\neq i}^{n} \hat{z}_t(M_{t.}\hat{z} + q_t) \geqq -(n-1)2^{-\mathbf{10L}}(2^{\mathbf{2L+1}} + 2^{\mathbf{L}}) > -2^{-(\mathbf{6L+1})}$.

□

Theorem 8.21 *Let $\varepsilon_0 = 2^{-(\mathbf{6L+1})}$. If $\mathbf{K} \neq \emptyset$, the n-dimensional volume of $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$ is $\geqq 2^{-\mathbf{11nL}}$.*

Proof. Assume $\mathbf{K} \neq \emptyset$. So (q, M) has a solution. Let $(\overline{w}, \bar{z})$ be a complementary BFS of (q, M). So, by Theorem 8.16, $\bar{z} \in \text{Bd}(\mathbf{E}) \cap \mathbf{K}$. For $\lambda > 0$ define the hypercube; $\mathbf{C}_\lambda = \{z : z \in \mathbf{R}^n, |z_j - \bar{z}_j| \leqq \frac{\lambda}{2}$ for all $j = 1$ to $n\}$. Then, clearly, the n-dimensional volume of $\mathbf{C}_\lambda$ is λ^n. We will now prove that $\mathbf{C}_\lambda \subset \mathbf{K}_1 \cap \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0}$ for $\lambda \leqq 2^{-\mathbf{11L}}$. Since the radius of $\mathbf{E}_0$ is $2^{\mathbf{L+1}}$, $\mathbf{C}_\lambda \subset \mathbf{E}_0$ by the definition of $\mathbf{C}_\lambda$ and the fact that $||\bar{z}|| < 2^{\mathbf{L}}$ from Theorem 8.18. Let $\hat{z}$ be any point in $\mathbf{C}_\lambda$. Since $\bar{z}_i \geqq 0$, $M_{i.}\bar{z} + q_i \geqq 0$ for all $i = 1$ to n, we have; $\hat{z}_i \geqq \bar{z}_i - \frac{\lambda}{2} \geqq -\frac{\lambda}{2} \geqq -2^{-\mathbf{10L}}$; $M_{i.}\hat{z} + q_i \geqq M_{i.}\bar{z} + q_i - \frac{\lambda}{2}\sum_{j=1}^{n} |m_{ij}| \geqq -2^{-(\mathbf{11L+1})} \times$

$2^{\mathbf{L}} \geqq -2^{-\mathbf{10L}}$. So $\mathbf{C}_\lambda \subset \mathbf{K}_1$. Also, since $\bar{z}^T(M\bar{z}+q) = 0$ (since $(\overline{w} = M\bar{z}+q, \bar{z})$ solves (q, M)), we have: $\hat{z}^T(M\hat{z}+q) = (\hat{z}-\bar{z})^T(M\bar{z}+q+M^T\bar{z}) + (\hat{z}-\bar{z})^T M(\hat{z}-\bar{z}) \leqq \frac{\lambda}{2}n(2^{\mathbf{L}} + 2^{\mathbf{L}}2^{\mathbf{L}}) + (\frac{\lambda}{2})^{\mathbf{2}} \sum_{i,j} |m_{ij}| \leqq 2^{-(\mathbf{11L+1})}n2^{\mathbf{2L+2}} + n^{\mathbf{2}}2^{\mathbf{L}-\mathbf{2(11L+1)}} \leqq \varepsilon_0$. This implies that $\mathbf{C}_\lambda \subset \mathbf{E}_{\varepsilon_0}$. Hence $\mathbf{C}_\lambda \subset \mathbf{K}_1 \cap \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0}$. Now letting $\lambda = 2^{-\mathbf{11L}}$, the volume of $\mathbf{C}_\lambda$ is $2^{-\mathbf{11L}}$, and these facts imply the theorem.

□

Let $\hat{z}$ be any point in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$. Define

$$\mathbf{J}_1^- = \{i : M_{i.}\hat{z} + q_i \leqq 0\}\ , \quad \mathbf{J}_1^+ = \{i : 0 < M_{i.}\hat{z} + q_i \leqq 2^{-\mathbf{3L}}\}\ ,$$
$$\mathbf{J}_2^- = \{i : \hat{z}_i \leqq 0\}\ , \quad \mathbf{J}_2^+ = \{i : 0 < \hat{z}_i \leqq 2^{-\mathbf{3L}}\}\ .$$

Then by Theorem 8.20, $\mathbf{J}_1^- \cup \mathbf{J}_1^+ \cup \mathbf{J}_2^- \cup \mathbf{J}_2^+ = \{1, \ldots, n\}$. Furthermore, $\hat{z}$ is a solution of :

$$\begin{aligned} -M_{i.}z &\leqq \quad q_i + 2^{-\mathbf{3L}}, \ i = 1 \text{ to } n \\ -z_i &\leqq \quad 2^{-\mathbf{3L}}, \qquad i = 1 \text{ to } n \\ M_{i.}z &\leqq -q_i + 2^{-\mathbf{3L}}, \text{ for } i \in \mathbf{J}_1^+ \\ z_i &\leqq \quad 2^{-\mathbf{3L}}, \qquad \text{for } i \in \mathbf{J}_2^+ \end{aligned} \tag{8.14}$$

Theorem 8.22 *Let $\hat{z}$ be any point in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$. Let I be the unit matrix of order n. Using the constructive procedure described by P. Gács and L. Lovász in [8.11] (see also Theorem 15.7, Chapter 15 of [2.26]) obtain a new solution, which we will denote by the same symbol $\hat{z}$, such that if $\mathbf{J}_1^-$, $\mathbf{J}_1^+$, $\mathbf{J}_2^-$, $\mathbf{J}_2^+$ are the index sets corresponding to this new $\hat{z}$, then the new $\hat{z}$ also satisfies (8.14), and there exists a linearly independent subset, $\mathbf{D} \subset \{M_{i.} : i \in \mathbf{J}_1^- \cup \mathbf{J}_1^+\} \cup \{I_{i.} : i \in \mathbf{J}_2^- \cup \mathbf{J}_2^+\}$ such that $\mathbf{D}$ spans linearly $\{M_{i.} : i = 1 \text{ to } n\} \cup \{I_{i.} : i = 1 \text{ to } n\}$. Furthermore, if $\bar{z}$ is a solution of :*

$$\begin{aligned} -M_{i.}z &= q_i, \ \textit{for } i \textit{ such that } M_{i.} \in \mathbf{D} \\ z_i &= 0, \ \ \textit{for } i \textit{ such that } I_{i.} \in \mathbf{D} \end{aligned}$$

then $(\overline{w} = M\bar{z}+q, \bar{z})$ is a solution of the LCP (q, M).

Proof. This theorem follows from the results of P. Gács and L. Lovász in [8.11] (or Theorem 15.7, Chapter 15 in [2.26]) applied on (8.14). We know that $\hat{z}$ satisfies :

$$\begin{aligned} -M_{i.}\hat{z} &\geqq \quad q_i, \quad \text{for } i \in \mathbf{J}_1^- \\ M_{i.}\hat{z} &\geqq -q_i, \quad \text{for } i \in \mathbf{J}_1^+ \\ -\hat{z}_i &\geqq \quad 0, \quad \text{for } i \in \mathbf{J}_2^- \\ \hat{z}_i &\geqq \quad 0, \quad \text{for } i \in \mathbf{J}_2^+ \end{aligned}$$

By these results, $\bar{z}$ is a solution of

$$\begin{aligned} -Mz &\leqq q \\ -z &\leqq 0\ . \end{aligned}$$

Furthermore, $\bar{z}$ satisfies :

$$\begin{aligned} M_{i.}\bar{z} &= -q_i, \quad \text{for } i \in \mathbf{J}_1^- \cup \mathbf{J}_1^+ \\ \bar{z}_i &= \quad 0, \quad \text{for } i \in \mathbf{J}_2^- \cup \mathbf{J}_2^+ \end{aligned}$$

by the spanning property of $\mathbf{D}$ and these results. Also, since $\{1, \dots, n\}$ is the union of $\mathbf{J}_1^-$, $\mathbf{J}_1^+$, $\mathbf{J}_2^-$, $\mathbf{J}_2^+$, at least one of w_i or z_i is zero for each $i = 1$ to n. All these facts together clearly imply that $(\overline{w}, \bar{z})$ is a solution of the LCP (q, M).

□

The Algorithm

Apply the ellipsoid algorithm discussed in Section 8.5 to get a point $\hat{z}$ in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$, initiating the algorithm with $z^0 = 0$, $A_0 = 2^{\mathbf{2(L+1)}} I$, $\mathbf{E}_0 = \mathbf{E}(z^0, A_0)$. In this case $\mathbf{K}$ could be $\emptyset$. This could be recognized in the ellipsoid algorithm in two different ways. For any r, if the quantity γ_r in step r of the ellipsoid algorithm turns out to be $\leqq -1$, it is an indication that the set $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1 = \emptyset$, terminate, in this case $\mathbf{K} = \emptyset$ and the LCP (q, M) has no solution (for a proof of this see Chapter 15 of [2.26]). If $\gamma_r > -1$, compute x^{r+1}, A_{r+1} and continue. The volume of $\mathbf{E}_0$ here is $< 2^{\mathbf{2n(L+1)}}$, and if $\mathbf{K} \neq \emptyset$, the volume of $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$ is $> 2^{-\mathbf{11nL}}$ by Theorem 8.21. Hence if $\mathbf{K} \neq \emptyset$, this ellipsoid algorithm will terminate in at most $2(n+1)^{\mathbf{2}}(13L+1)$ steps with a point $\hat{z} \in \mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$. So, if the ellipsoid algorithm did not find a point in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$ even after $2(n+1)^{\mathbf{2}}(13L+1)$ steps, we can conclude that $\mathbf{K} = \emptyset$, that is, that the LCP (q, M) has no solution. On the other hand, if a point $\hat{z}$ in $\mathbf{E}_0 \cap \mathbf{E}_{\varepsilon_0} \cap \mathbf{K}_1$ is obtained in the ellipsoid algorithm, then using it, obtain a solution $(\overline{w}, \bar{z})$ of the LCP (q, M) as discussed in Theorem 8.22.

8.7 Some NP-Complete Classes of LCPs

The ellipsoid algorithm discussed in Section 8.4, 8.5, 8.6 can only process LCPs associated with PSD matrices (the class of these LCP is equivalent to the class of convex quadratic programs). In [8.6, 8.15] it was shown that certain LCPs satisfying special properties can be solved as linear programs, and these LCPs are therefore polynomially solvable using the ellipsoid algorithm (see Chapter 15 in [2.26]) on the resulting linear programs.

For the general LCP, the prospects of finding a polynomially bounded algorithm are not very promising, in view of the result in [8.3] where it is shown that this problem is $\mathcal{NP}$-complete. See reference [8.12] for the definition of $\mathcal{NP}$-completeness. Let $a_1, \dots,$ a_n, a_0 be positive integers and let M_{n+2} and $q(n+2)$ be the following matrices :

$$M_{n+2} = \begin{pmatrix} -I_n & 0 & 0 \\ e_n^T & -n & 0 \\ -e_n^T & 0 & -n \end{pmatrix} , \quad q(n+2) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ -a_0 \\ a_0 \end{pmatrix}$$

where I_n denotes the unit matrix of order n, and e_n is the column vector in $\mathbf{R}^n$ all of whose entries are 1. Also consider the 0-1 equality constrained Knapsack feasibility problem :

$$\begin{aligned} &\sum_{i=1}^{n} a_i x_i = a_0 \\ &x_i = 0 \text{ or } 1 \quad \text{for all } i = 1 \text{ to } n \ . \end{aligned} \tag{8.15}$$

If $(\widetilde{w}, \tilde{z})$ is a solution of the LCP $(q(n+2), M_{n+2})$, define $\tilde{x}_i = \frac{\tilde{z}_i}{a_i}$, $i = 1$ to n, and verify that $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ is a feasible solution of the Knapsack problem (8.15). Conversely of $\hat{x} = (\hat{x}_1, \dots, \hat{x}_r)^T$ is a feasible solution of (8.15), define $\widehat{w}_{n+1} = \hat{z}_{n+1} = \widehat{w}_{n+2} = \hat{z}_{n+2} = 0$ and $\hat{z}_i = a_i \hat{x}_i$, $\widehat{w}_i = a_i(1 - \hat{x}_i)$, $i = 1$ to n; and verify that $(\widehat{w} = (\widehat{w}_1, \dots, \widehat{w}_{n+2}), \hat{z} = (\hat{z}_1, \dots, \hat{z}_{n+2}))$ is a solution of the LCP $(q(n+2), M_{n+2})$. Since the problem of finding whether a feasible solution for (8.15) exists is a well known $\mathcal{NP}$-complete problem (see [8.12]), the problem of checking whether the LCP $(q(n+2), M_{n+2})$ has a solution is $\mathcal{NP}$-complete. Also, since the matrix M_{n+2} is negative definite, the class of LCPs associated with negative definite or negative semidefinite matrices are $\mathcal{NP}$-hard. Also M_{n+2} is lower triangular. This shows that the class of LCPs associated with lower or upper triangular matrices is $\mathcal{NP}$-hard, if negative entries appear in the main diagonal.

Let M be a given negative definite matrix with integer entries, and let $q \in \mathbf{R}^n$ be a given integer column vector. In this case the LCP (q, M) may not have a solution; and even if it does, the solution may not be unique. From the results in Chapter 3 we know that the number of distinct solutions of the LCP (q, M) in this case is finite. Define :

$$\mathbf{K} = \{z : \ z \geqq 0, \ Mz + q \geqq 0\}$$
$$\mathbf{E} = \{z : \ z^T(Mz + q) \geqq 0\}$$

Since M is negative definite, $\mathbf{E}$ is an ellipsoid. Let $\text{Bd}(\mathbf{E})$ = boundary of $\mathbf{E} = \{z : z^T(Mz + q) = 0\}$.

Clearly any point $z \in \text{Bd}(\mathbf{E}) \cap \mathbf{K}$ satisfies the property that $(w = Mz + q, z)$ is a solution of the LCP (q, M) and vice versa. So solving the LCP (q, M) is equivalent to the probem of finding a point in $\text{Bd}(\mathbf{E}) \cap \mathbf{K}$. However, in this case $\mathbf{K} \subset \mathbf{E}$, and in general, $\text{Bd}(\mathbf{E}) \cap \mathbf{K} \subset \mathbf{E} \cap \mathbf{K}$. See Figure 8.4. So the nice property that $\mathbf{E} \cap \mathbf{K} = \text{Bd}(\mathbf{E}) \cap \mathbf{K}$ which held for LCPs associated with PSD matrices does not hold here anymore, which makes the LCP associated with a negative definite matrix much harder. In this case (i. e., with M being negative definite), it is possible to find a point in $\mathbf{E} \cap \mathbf{K}$ using an ellipsoid algorithm (actually since $\mathbf{K} \subset \mathbf{E}$ here, a point in $\mathbf{K}$ can be found by the ellipsoid algorithm of Chapter 15 of [2.26] and that point will also lie in $\mathbf{E}$), but the point in $\mathbf{E} \cap \mathbf{K}$ obtained by the algorithm may not be on the boundary of $\mathbf{E}$, and hence may not lead to a solution of the LCP (q, M). In fact, finding a point in $\text{Bd}(\mathbf{E}) \cap \mathbf{K}$ is a concave minimization problem, and that's why it is $\mathcal{NP}$-hard.

The status of the LCPs (q, M) where M is a P-but not PSD matrix, is unresolved. In this case the LCP (q, M) is known to have a unique solution by the results in Chapter 3, but the sets $\{z : z^T(Mz + q) \leqq 0\}$ are not ellipsoids. The interesting question is whether a polynomially bounded algorithm exists for solving this special class of LCPs. This still remains an open question. It is also not known whether these LCPs are $\mathcal{NP}$-hard.

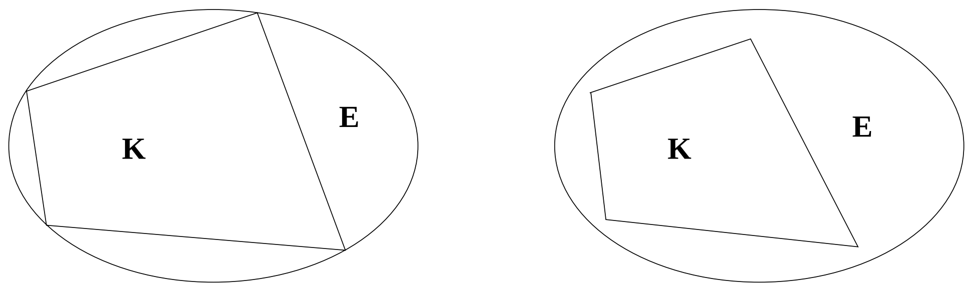

Figure 8.4 When M is negative definite, **E** and **K** may be as in one of the figures given here. Points of **K** on the boundary of **E**, if any, lead to solutions of the LCP (q, M).

8.8 An Ellipsoid Algorithm for Nonlinear Programming

In [8.9] J. Ecker and M. Kupferschmid discussed an application of the ellipsoid algorithm to solve NLPs of the following form :

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leqq 0, \quad i = 1 \text{ to } m \end{array}$$

where all the $f_i(x)$ are differentiable functions defined on $\mathbf{R}^n$, and we assume that $n > 1$.

For the convergence of the ellipsoid algorithm, we need to specify an initial ellipsoid whose intersection with a neighborhood of an optimum solution has positive n-dimensional volume. This requirement prevents the algorithm from being used in a simple way for problems having equality constraints, but the penalty transformation discussed in Section 2.7.6 can be used for them.

It is assumed that lower and upper bounds are avaible on each variable. l, u are these lower and upper bound vectors. The initial ellispoid is chosen to be the one of smallest volume among those ellipsoids with center $x^0 = \frac{l+u}{2}$ and containing $\{x : l \leqq x \leqq u\}$. Let this be $\mathbf{E}_0 = \{x : (x - x^0)^T D_0^{-1} (x - x^0) \leqq 1\} = \mathbf{E}_0(x^0, D_0)$, where

$$D_0 = \frac{n}{4} \begin{pmatrix} (u_1 - l_1)^2 & 0 & 0 & \dots & 0 \\ 0 & (u_2 - l_2)^2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (u_n - l_n)^2 \end{pmatrix} .$$

Suppose we have $\mathbf{E}_r(x^r, D_r)$. If x^r is infeasible, choose a violated constraint, say the i^{th}, where $f_i(x^r) > 0$. In case x^r is infeasible, the index i of the selected constraint is that of the first violated constraint encountered under a search of the constraints in cyclical order beginning with the constraint selected in the previous step. If x^r is feasible and $\nabla f_0(x^r) = 0$, terminate, x^r is optimal to NLP (under convexity assuptions, it is a stationary point otherwise). If x^r is feasible and $\nabla f_0(x^r) \neq 0$, choose the index i to be zero.

Having selected the index i (corresponding to a violated constraint if x^r is infeasible, or the objective function if x^r is feasible and $\nabla f_0(x^r) \neq 0$), let $\mathbf{H}_r$ be the hyperplane

$$\mathbf{H}_r = \{x : \ -(\nabla f_i(x^r))(x - x^r) = 0\} \ .$$

The hyperplane $\mathbf{H}_r$ supports the contour $f_i(x) = f_i(x^r)$ and divides the ellipsoid in half. The center x^{r+1} of the next ellipsoid $\mathbf{E}_{r+1}$ and the PD matrix D_{r+1} used in defining $\mathbf{E}_{r+1}$ are determined by the updating formulae

$$\begin{aligned} h &= \frac{\nabla f_i(x^r)}{||\nabla f_i(x^r)||} \\ d &= \frac{-D_r h^T}{+\sqrt{hD_r h^T}} \\ x^{r+1} &= x^r + \frac{d}{n+1} \\ D_{r+1} &= \frac{n^2}{n^2 - 1}\Big(D_r - \frac{2}{n+1} dd^T\Big) \ . \end{aligned}$$

The best point obtained during the algorithm and its objective value are maintained. Various stopping rules can be employed, such as requiring the difference between successive best values to be sufficiently small, etc.

The method is best suited for solving the NLP above, when all the functions $f_i(x)$ are convex. If a nonconvex function is used to generate the hyperplane H_r that cuts $\mathbf{E}_r$ in half, the next ellipsoid may not contain the optimal point, and the algorithm may converge to a point that is not even stationary.

In computational tests carried out by J. G. Ecker and M. Kupferschmid [8.9], this method performed very well.

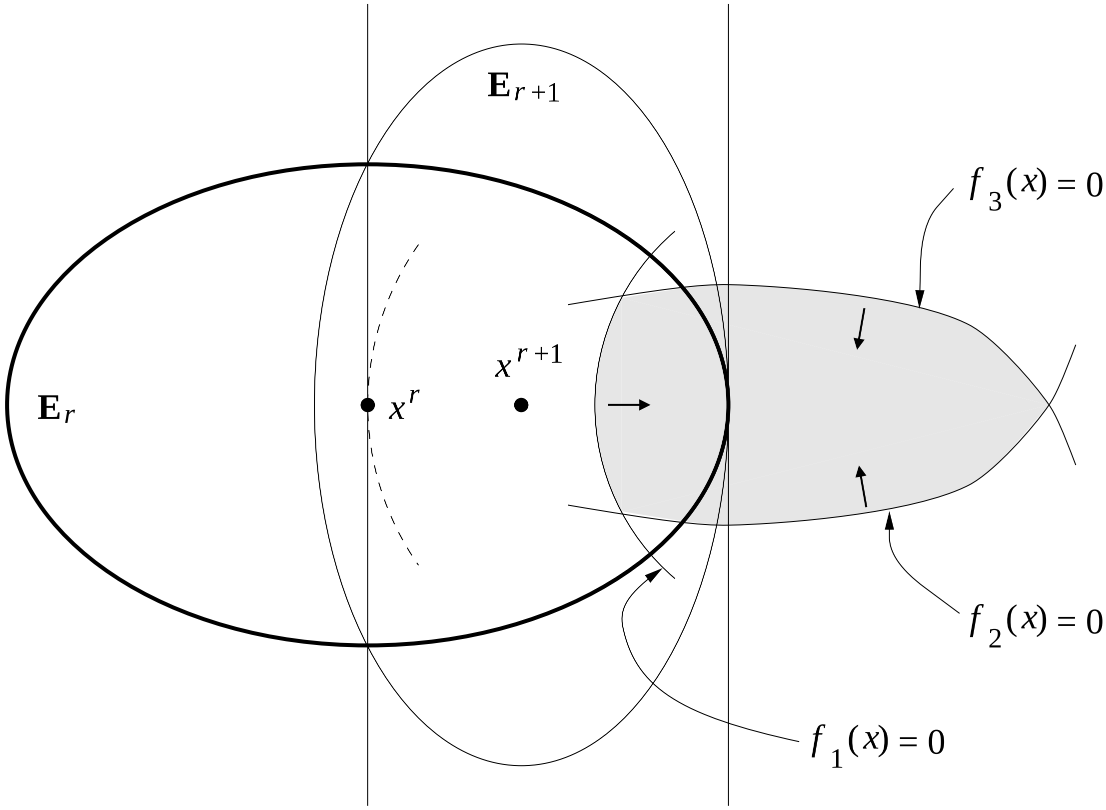

Figure 8.5 Construction of the new ellipsoid when x^r is infeasible. The arrow on constraint surface $f_i(x) = 0$ indicates the feasible side, that is satisfying $f_i(x) \leqq 0$. $f_1(x) \leqq 0$ is violated at x^r and is selected.

8.9 Exercises

8.4 Let A, D, b, d be given matrices of orders $m_1 \times n$, $m_2 \times n$, $m_1 \times 1$, $m_2 \times 1$ respectively with integer entries. Let F be a given PD symmetric matrix of order n with integer entries. Define.

$$\mathbf{K}_1 = \{x : \; Ax \geqq b\}$$
$$\mathbf{K}_2 = \{x : \; Dx \geqq d\}$$
$$\mathbf{E} = \{x : \; x^T F x \leqq 1\} \; .$$

Construct polynomially bounded algorithms for checking whether

(i) $\mathbf{K}_1 \subset \mathbf{K}_2$

(ii) $\mathbf{E} \subset \mathbf{K}_1$.

Does a polynomially bounded algorithm exists for checking whether $\mathbf{K}_1 \subset \mathbf{E}$? Why ?

8.5 Consider the quadratic program

$$\begin{array}{ll} \text{minimize} & cx + \frac{1}{2} x^T D x \\ \text{subject to} & x \leqq b \end{array}$$

where $b > 0$ and D is a Z-matrix of order n. Express the KKT optimality conditions for this problem in the form of a special type of linear complementarity problem, and develop a special direct method for solving it, based on Chandrasekaran's algorithm discussed in Section 8.1.
(J. S. Pang [8.17])

8.6 Study the computational complexity of the problem of checking whether the ellipsoid $\mathbf{E} = \{x : (x - \bar{x})^T D(x - \bar{x}) \leqq 1\}$ where D is given integer PD symmetric matrix and $\bar{x}$ is a given noninteger rational point, contains an integer point.

8.7 Show that the LCP (q, M) is equivalent to the following piecewise linear concave function minimization problem.

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} (\text{minimum}\{0, M_{j.}z - z_j + q_j\} + z_j) \\ \text{subject to} & Mz + q \geqq 0 \\ & \quad\;\; z \geqq 0. \end{array}$$

(O. L. Mangasarian)

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