One-Dimensional Liquid Fibers

W. W. SCHULTZ and S. H. DAVIS, *Department of Engineering Sciences and Applied Mathematics, The Technological Institute, Northwestern University, Evanston, Illinois 60201*

Synopsis

Axisymmetric Newtonian viscous fibers are examined under isothermal conditions. An expansion based on lubrication scaling is used to derive systematically the one-dimensional equations for the fiber. At lowest order these are identical with those obtained by others, e.g., Matovich and Pearson. One-dimensional theory limitations on jet shape and inertial, gravitational, and surface tension effects are obtained from higher-order approximations. Similar approaches can be used to analyze fibers having complex rheological behaviors.

INTRODUCTION

Slender axisymmetric liquid jets have important applications in fluid mechanics. High-inertia jets are used in rock cutting. Surface-tension-dominated jets are used in printing. Fluid/boundary interactions dominate fluidic logical devices. In the present work we address fiber-forming configurations where viscous forces dominate. In textile applications molten glass or polymer leaves an orifice and rapidly hardens due to heat or mass transfer to the environment or due to chemical reaction. The fiber is then "wound-up" downstream, the winder imposing a fixed velocity at the fiber end. In other applications, large body forces in the axial direction stretch the jet. In this case the fiber end is force free. Although there are some numerical[1,2] and experimental[3,4] studies of such viscous flows near the orifices, most of the analysis applies away from the orifices where the flow is taken to be one dimensional.[5–8] By this, the authors mean that the axial velocity and pressure are independent of the radial coordinate or that the axial velocity has become rectilinear.

In his review of this subject, Denn[9] credits Matovich and Pearson[7] with the "most careful derivation" of the one-dimensional equations. Although Matovich and Pearson do set up an expansion procedure,

Journal of Rheology, 26(4), 331–345 (1982) CCC 0148-6055/82/040331-15$01.50

they do not define a small parameter for the expansion. Instead, they formulate directly a one-dimensional momentum balance in the same sense as Glicksman.[6] To do this one *assumes* that the pressure and axial velocity are independent of the radial coordinate. Geyling[10] has attempted to use a parametric expansion for the textile fiber process but it is not clear that he satisfies all the relevant boundary conditions. He does not calculate corrections to the one-dimensional approximation.

There have been attempts at the analysis of jets by the use of coordinate expansions. Kase[11] expands axial velocity in powers of the radial coordinate. Kaye and Vale[12] and Clarke[8] essentially expand in inverse powers of the axial coordinate. Unlike parametric expansions, these coordinate expansions cannot satisfy conditions at all the fiber boundaries.

In the present study we consider the simple fiber system, namely, a steady, axisymmetric jet of a Newtonian, constant-property liquid emptying into a passive gaseous atmosphere. Viscous, inertial, gravitational, and surface tension forces are included.

We formalize the notion that the jet is slender by applying lubrication scaling ideas and develop the solutions in powers of ϵ, a slenderness ratio. At leading order we obtain the one-dimensional system identical to that of previous authors (e.g., Matovich and Pearson[7]). However, since our procedure is a parametric expansion, we can obtain higher-order corrections to the one-dimensional theory. These corrections show that axial velocity and the pressure depend on the radial coordinate. Having obtained these, we can estimate the validity of the one-dimensional theory. We find that the limitations for the textile configurations are quite modest, but that the limitation on the effect of gravity in some processes can be quite severe.

Thus, the two objects of this work are (i) the systematic derivation of the one-dimensional approximations from the full axisymmetric problem and (ii) the estimation of the validity of the one-dimensional theory by examining higher-order corrections. Clearly, such procedures are applicable to the more complicated systems as well.

The asymptotic procedure we use gives an "outer" solution to the problem valid in regions sufficiently far from either ends of the fiber. The "inner" solutions valid at the ends are boundary-layer corrections, which in the present case are solutions to the full governing equations. Since these can only be solved using extensive numerical analysis, we bypass this and pose "average" boundary conditions at the ends. The numerical analysis by Fisher, Denn, and Tanner[2] indicates that the boundary-layer corrections are confined to within one diameter of the orifice. These averaged conditions are imposed on the outer variables and substantially coincide with those conditions previous authors have used.

FORMULATION FOR THE FIBER WITH A WINDER

A slender, axisymmetric liquid jet (fiber) steadily emerges from an orifice of radius r_I into a passive gaseous environment. At a distance L the fiber is "wound up" at a given average axial velocity w_W. The system is isothermal and gravity acts in the axial direction. The jet liquid is Newtonian, all fluid properties are constant including the surface tension σ on the liquid–gas interface. A cylindrical coordinate system is employed as shown in Figure 1.

We scale the governing system of equations and boundary conditions consistent with a fiber whose flow properties vary slowly in the axial direction. We scale the axial and radial coordinates on L and r_I, respectively. In order to preserve conservation of mass, we scale the axial and radial velocities on w_I and $w_I r_I/L$, respectively, where w_I is the average axial speed through the orifice. Pressure is scaled

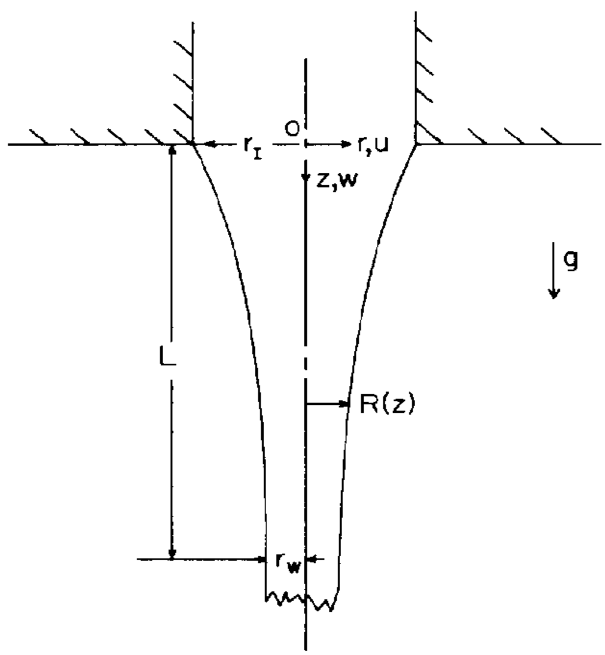

Fig. 1. Schematic figure of liquid fiber showing the coordinate system and notation.

on $\mu w_I/L$ so that the radial momentum equation balances the pressure gradient and the viscous forces. (Using the axial momentum balance between pressure gradient and viscous forces would result in the same results differently named.)

The resulting nondimensional continuity and Navier–Stokes equations are as follows:

$$u_r + (1/r)u + w_z = 0, \tag{1a}$$

$$\epsilon^2 \mathrm{Re}(uu_r + wu_z) = -p_r + u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + \epsilon^2 u_{zz}, \tag{1b}$$

and

$$\epsilon \mathrm{Re}(uw_r + ww_z) = -\epsilon^2 p_z + w_{rr} + \frac{1}{r}w_r + \epsilon^2 w_{zz} + \frac{\mathrm{Re}}{\mathrm{Fr}}. \tag{1c}$$

The interface between the liquid and the gas is located at $r = R(z)$. The kinematic boundary condition is given by

$$u = wR_z \quad \text{on } r = R(z). \tag{2a}$$

On the interface, the shear stress is zero,

$$2\epsilon^2 R_z(u_r - w_z) + (1 - \epsilon^2 R_z^2)(\epsilon^2 u_z + w_r) = 0 \quad \text{on } r = R(z), \tag{2b}$$

and the normal stress is balanced by surface tension times curvature,

$$\epsilon \mathrm{Ca}\{p(1 + \epsilon^2 R_z^2) - 2[u_r + \epsilon^2 R_z^2 w_z - R_z(\epsilon^2 u_z + w_r)]\}$$
$$= \frac{1}{R}(1 + \epsilon^2 R_z^2)^{1/2} - \epsilon^2 R_{zz}(1 + \epsilon^2 R_z^2)^{-1/2} \quad \text{on } r = R(z). \tag{2c}$$

On the axis of the fiber, all physical quantities must be bounded,

$$|u|, |w|, |p| < \infty \quad \text{on } r = 0. \tag{3}$$

At the orifice, the liquid attaches to the sharp corner, and both the velocity field and the diameter of the fiber are given,

$$u = \hat{u}_I(r), \quad w = \hat{w}_I(r), \quad R = 1 \quad \text{on } z = 0. \tag{4a}$$

Finally, the fiber is "wound up" at its end, so that the velocity field is known there:

$$u = \hat{u}_W(r), \quad w = \hat{w}_W(r) \quad \text{on } z = 1. \tag{4b}$$

In conditions (4a) and (4b), the cross-sectional averages of $\hat{w}_I$ and $\hat{w}_W$

are, respectively, w_I and w_W. In the above system subscripts r and z denote partial differentiation. The five parameters are

Reynolds number	$\mathrm{Re} = w_I r_I/\nu$,	(5a)
Froude number	$\mathrm{Fr} = w_I^2/g r_I$,	(5b)
capillary number	$\mathrm{Ca} = w_I \mu/\sigma$,	(5c)
log of the extension ratio	$\alpha = \ln(w_W/w_I)$,	(5d)
scaling parameter	$\epsilon = r_I/L$.	(5e)

In the above g is the gravitational acceleration; μ and ν are the dynamic and kinematic liquid viscosities, respectively; and σ is the surface tension coefficient of the liquid–gas interface.

We shall seek asymptotic solutions for slender jets having $\epsilon \to 0$ with all other parameters fixed.

VISCOUS-ONLY PROBLEM: LEADING-ORDER SOLUTIONS

We first consider the case when inertial, gravitional, and surface tension forces are negligible, i.e., we set $\mathrm{Re} = 0$, $\mathrm{Fr}^{-1} = 0$, and $\mathrm{Ca}^{-1} = 0$ in Eqs. (1)–(3).

We suppose that α is fixed and ϵ is small and write for all dependent variables φ,

$$\varphi = \varphi_0 + \epsilon^2\varphi_2 + \epsilon^4\varphi_4 + O(\epsilon^6), \tag{6}$$

where we have noted that for this case the small parameter ϵ appears only in even powers.

If forms (6) are substituted into Eqs. (1)–(3), we obtain at order unity in ϵ,

$$u_{0_r} + \frac{1}{r}u_0 + w_{0_z} = 0, \tag{7a}$$

$$u_{0_{rr}} + \frac{1}{r}u_{0_r} - \frac{1}{r^2}u_0 - p_{0_r} = 0, \tag{7b}$$

$$w_{0_{rr}} + \frac{1}{r}w_{0_r} = 0, \tag{7c}$$

with boundary conditions,

$$u_0 - w_0 R_{0_z} = 0 \qquad \text{on } r = R_0, \tag{8a}$$

$$w_{0_r} = 0 \qquad \text{on } r = R_0, \tag{8b}$$

$$p_0 - 2(u_{0_r} - R_{0_z} w_{0_r}) = 0 \qquad \text{on } r = R_0, \tag{8c}$$

$$|u_0|, |w_0|, |p_0| < \infty \qquad \text{on } r = 0. \tag{9}$$

The interfacial conditions (8) are obtained by expanding the boundary conditions (2) about $\epsilon = 0$ and so evaluating the interfacial conditions at the position R_0.

As mentioned earlier, the end conditions (4) can be satisfied by posing boundary layers at $z = 0$ and $z = 1$ and matching. This gives rise to equivalent conditions at $z = 0$ and $z = 1$ on the outer variables u_0, w_0, p_0, and R_0. However, the boundary-layer equations appropriate here are essentially the full governing equations which cannot be solved except by numerical means. We thus reformulate the end conditions (4) by stating conditions that can be posed on the outer equations. We shall regard the end conditions as statements on the cross-sectional average values of w. Thus, we write

$$\frac{2}{R^2(z)} \int_0^{R(z)} rw(r,z)dr = 1 \quad \text{on } z = 0, \tag{10a}$$

with

$$R(z) = 1 \quad \text{on } z = 0, \tag{10b}$$

and

$$\frac{2}{R^2(z)} \int_0^{R(z)} rw(r,z)dr = e^{\alpha} \quad \text{on } z = 1. \tag{10c}$$

We have thus abandoned the pointwise conditions on u and w in (4) since such conditions are determined by matching. This procedure has been used by all previous authors though they have not stated it in the same way.

If the expansion (6) is substituted into forms (10), we obtain at order unity in ϵ,

$$2 \int_0^1 rw_0 dr = R_0 = 1 \quad \text{on } z = 0, \tag{11a}$$

$$\frac{2}{R_0^2} \int_0^{R_0} rw_0 dr = e^{\alpha} \quad \text{on } z = 1. \tag{11b}$$

We now find the leading-order solution. Equation (7c) and the boundedness condition (9) give

$$w_0 = W_0(z), \tag{12}$$

where W_0 is an arbitrary function at this stage. If we solve continuity (7a) for u_0 and use boundedness (9) and form (12), we find that

$$u_0 = -(1/2)rW_0'(z), \tag{13}$$

where a prime denotes differentiation. Then, using forms (12) and (13), we can solve for the pressure from Eq. (7b) and boundary condition (8c),

$$p_0 = -W_0'(z). \tag{14}$$

At this point, the analysis shows that to leading order in ϵ the axial velocity and pressure are independent of r. These are the two main assumptions used by Matovich and Pearson,[7] Glicksman,[6] and others to develop the one-dimensional equations. Therefore, our leading-order equations will be identical to those of others for the present case: $\mathrm{Re} = \mathrm{Fr}^{-1} = \mathrm{Ca}^{-1} = 0$.

We use forms (12) and (13), the end condition (11a), and the kinematic condition (8) to find that

$$W_0R_0^2 = 1. \tag{15}$$

In order to determine W_0, we proceed to the order ϵ^2 problem. This takes the form

$$u_{2_r} + \frac{1}{r}u_2 + w_{2_z} = 0, \tag{16a}$$

$$u_{2_{rr}} + \frac{1}{r}u_{2_r} - \frac{1}{r^2}u_2 - p_{2_r} = -u_{0_{zz}}, \tag{16b}$$

$$w_{2_{rr}} + \frac{1}{r}w_{2_r} = p_{0_z} - w_{0_{zz}}, \tag{16c}$$

with boundary conditions,

$$u_2 - w_2R_{0_z} = R_{2_z}w_0 - R_2u_{0_r} \qquad \text{on } r = R_0, \tag{17a}$$

$$w_{2_r} = 2R_{0_z}(w_{0_z} - u_{0_r}) - u_{0_z} \qquad \text{on } r = R_0, \tag{17b}$$

$$p_2 - 2(u_{2_r} - R_{0_z}w_{2_r}) = -p_0R_{0_z}^2 + 2R_{0_z}(R_{0_z}w_{0_z} - u_{0_z}) \qquad \text{on } r = R_0, \tag{17c}$$

$$|u_2|, |w_2|, |p_2| < \infty \qquad \text{on } r = 0. \tag{18}$$

Interfacial conditions (17) have been simplified using solution (12). The end conditions (10) become at this order

$$\frac{2}{R_0^2}\int_0^{R_0} rw_2 dr = 0 \qquad \text{on } z = 0,1, \tag{19a}$$

and

$$R_2 = 0 \quad \text{on } z = 0. \tag{19b}$$

We solve for w_2 from (16c) using forms (12) and (14) and boundedness (18):

$$w_2 = -(1/2)r^2 W_0'' + W_2(z), \tag{20}$$

where W_2 is an arbitrary function. Form (20) represents the first radially dependent axial velocity. We now substitute forms (12), (13), and (20) into the shear-stress boundary condition (17b) to obtain

$$R_0 W_0'' + 2R_0' W_0' = 0. \tag{21}$$

We thus have a pair of ordinary differential equations, (15) and (21), that determine both R_0 and W_0. If we eliminate R_0 between these, we have

$$W_0'' W_0 = W_0'^2 \tag{22a}$$

subject to

$$W_0(0) = 1 \tag{22b}$$

and

$$W_0(1) = e^{\alpha}. \tag{22c}$$

Conditions (22b) and (22c) are the simplified forms of (11a) and (11b) using the fact that w_0 depends on z only.

This is the standard system given in Matovich and Pearson,[7] governing steady, one-dimensional, isothermal fiber flow in the (Newtonian) viscosity-dominated regime. The solutions are

$$W_0 = e^{\alpha z} \tag{23a}$$

and

$$R_0 = e^{-(1/2)\alpha z}. \tag{23b}$$

We thus see that the fiber becomes exponentially more slender (while the speed increases) as one moves from the orifice toward the winder.

VISCOUS-ONLY PROBLEMS: HIGHER-ORDER CORRECTIONS

Given the simple form of the leading-order solutions, we can obtain corrections relatively easily.

We solve continuity (16a) by using form (20) and boundedness (18):

$$u_2 = (1/8)\alpha^3 r^3 W_0 - (1/2) r W_2'. \tag{24}$$

If we use forms (13) and (24), and use the normal stress condition (17c), we can find the pressure correction:

$$p_2 = -(1/4)\alpha^3(r^2 W_0 - 1) - W_2', \tag{25}$$

where we have used the solutions (23). The kinematic boundary condition (17a) simplifies to yield,

$$R_2 = \left(\frac{1}{8}\alpha^2 - \frac{1}{2}W_2\right)e^{-(3/2)\alpha z}, \tag{26}$$

where we again have used solution (23).

In order to determine W_2 and R_2, we need to examine the order ϵ^4 equations. We shall not give the details here, but straightforward analysis gives

$$W_2 = (1/8)\alpha^2[1 - z(1 - e^{-\alpha})]e^{\alpha z} + (1/8)\alpha^2 \tag{27a}$$

and

$$R_2 = (1/16)\alpha^2[e^{-\alpha z} - 1 + z(1 - e^{-\alpha})]e^{-(1/2)\alpha z}, \tag{27b}$$

where we have used $W_2(0) = W_2(1) = \alpha^2/4$ obtained from conditions (19a) and (19b).

It can be shown easily that $R_2 \leq 0$ when $\alpha > 0$ (i.e., for an accelerating jet). Hence, *the leading-order solution overestimates the fiber diameter* and neglects radial variations of pressure and axial velocity. A summary of the appropriate solutions follows:

$$w = e^{\alpha z}\left\{1 + \epsilon^2\alpha^2\left[-\frac{1}{2}r^2 + \frac{1}{8}(1 - z + ze^{-\alpha} + e^{-\alpha z})\right] + O(\epsilon^4\alpha^4)\right\}, \tag{28a}$$

$$R = e^{-(1/2)\alpha z}\left\{1 + \frac{1}{16}\epsilon^2\alpha^2(e^{-\alpha z} - 1 + z - ze^{-\alpha}) + O(\epsilon^4\alpha^4)\right\}, \tag{28b}$$

$$u = -\frac{1}{2}\alpha r e^{\alpha z}\left\{1 + \epsilon^2\alpha^2\left[-\frac{1}{4}r^2 + \frac{1}{8}\left(1 - z + ze^{-\alpha} - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right)\right] + O(\epsilon^4\alpha^4)\right\}, \quad (28c)$$

$$p = -\alpha e^{\alpha z}\left\{1 + \epsilon^2\alpha^2\left[\frac{1}{4}(e^{-\alpha z} - r^2) + \frac{1}{8}\left(1 - z + ze^{-\alpha} - \frac{1}{\alpha} + \frac{1}{\alpha}e^{-\alpha}\right)\right] + O(\epsilon^4\alpha^4)\right\}. \quad (28d)$$

CRITERION FOR VALIDITY OF ONE-DIMENSIONAL SOLUTIONS

One benefit of using an explicit perturbation procedure to deduce the one-dimensional equations from the axisymmetric ones is the capacity to calculate higher-order corrections. These can then be used to determine the conditions under which the one-dimensional solutions are an adequate description of the flow.

We wish to estimate when the order ϵ^2 corrections to the one-dimensional solutions are negligible. Clearly, many criteria are possible. We shall use a criterion insuring that the fiber axial-velocity profile is sufficiently flat; this occurs when

$$\frac{|w[R(z),z] - w(0,z)|}{|w(0,z)|} \ll 1. \quad (29)$$

If we use the two-term solution (28a) to evaluate w, and note that w_0 is independent of r, then inequality (29) measures the r-dependent w_2 in terms of w_0. This estimate has the form

$$(1/2)\epsilon^2 R_0^2 W_0'' \ll W_0, \quad (30)$$

or alternatively,

$$\epsilon^2\alpha^2 \ll 2, \quad (31)$$

since max $R_0 = 1$. We thus see that the natural small parameter in this problem is not ϵ^2 itself but $\epsilon^2\alpha^2$ as is emphasized in forms (28). Many calculations have shown that $R_0 \gg \epsilon^2 R_2$ when the "stricter" criterion (31) is satisfied. In fact the steady solution (23) is stable (see, e.g., ref. 13) only for $\alpha < 3.01$. Thus, whenever the steady fiber flow is stable, criterion (31) is

$$\epsilon \ll 0.47. \quad (32a)$$

If we write the *dimensional* interfacial position as $\hat{r} = \hat{R}(\hat{z})$, then the slope $d\hat{R}/d\hat{z}$ is measured by ϵ and condition (32a) translates to

$$\max \left| \frac{d\hat{R}}{d\hat{z}} \right| \ll 0.71. \tag{32b}$$

Condition (32) also guarantees $\epsilon^2 R_2 \ll R_0$ for a stable fiber. Conditions (32) are, indeed, not very limiting; the one-dimensional solutions are normally excellent approximations.

GRAVITATIONAL, INERTIAL, AND SURFACE TENSION EFFECTS: LEADING ORDER

We wish now to include gravity, inertia, and surface tension in the fiber flow expansion. To be consistent with small ϵ, these quantities must be limited in precise ways.

If we include gravitational effects in the fiber and $\mathrm{Re}/\mathrm{Fr} = O(1)$ as $\epsilon \to 0$ in Eq. (1c), then w_0 of the modified equation (7c) would be parabolic in r. However, this form would be incompatible with the shear-stress boundary condition (8b). If, instead, $\mathrm{Re}/\mathrm{Fr} = O(\epsilon)$ as $\epsilon \to 0$ in Eq. (1c), expansion (6) would be replaced by an expansion in powers of ϵ. This time the $O(\epsilon)$ shear-stress boundary condition could not be satisfied. It turns out that gravity can be included in the analysis only if $\mathrm{Re}/\mathrm{Fr} = O(\epsilon^2)$ so we write

$$G \equiv \mathrm{Re}/\epsilon^2\mathrm{Fr} = O(1) \text{ as } \epsilon \to 0. \tag{33}$$

Likewise, if we wish to include inertia and surface tension in the present formulation, we must assume that

$$\overline{\mathrm{Re}} \equiv \mathrm{Re}/\epsilon = O(1) \text{ as } \epsilon \to 0 \tag{34}$$

and

$$\overline{\mathrm{Ca}} \equiv \epsilon\mathrm{Ca} = O(1) \text{ as } \epsilon \to 0. \tag{35}$$

Thus, if forms (33)–(35) are substituted into system (1)–(3) and expansion (6) assumed, then we can develop solutions in powers of ϵ^2.

The only change at leading order is that the normal stress boundary condition (8c) is replaced by a surface-tension-dependent form,

$$p_0 - 2(u_{0_r} - R_{0_z} w_{0_r}) = (\overline{\mathrm{Ca}}\, R_0)^{-1} \text{ on } \quad r = R_0. \tag{36}$$

By making gravity and inertia small according to assumptions (33) and (34), we ensure that these effects are postponed to the order ϵ^2 equations. Thus, w_0 is still independent of r, $w_0 = W_0(z)$, and the order ϵ^2 axial momentum equation (16c) is replaced by

$$w_{2_{rr}} + (1/r)w_{2_r} = p_{0_z} - w_{0_{zz}} - G + \overline{\mathrm{Re}}\, w_0 w_{0_z} \tag{37}$$

whose solution, subject to the appropriate boundary conditions, takes the form

$$w_2 = \frac{1}{4} r^2 \left(\overline{\mathrm{Re}}\, W_0 W_0' - G - 2W_0'' + \frac{1}{2} \overline{\mathrm{Ca}}^{-1}\, W_0' W_0^{-1/2} \right) + W_2(z). \tag{38}$$

If result (38) is substituted into shear-stress boundary condition (17b) and R_0 is eliminated using result (15), we obtain a single ordinary differential equation for W_0,

$$3(W_0'^2 W_0^{-1} - W_0'') + \frac{1}{2} \overline{\mathrm{Ca}}^{-1}\, W_0' W_0^{-1/2} + \overline{\mathrm{Re}}\, W_0 W_0' - G = 0 \tag{39a}$$

subject to the same end conditions as before, viz.,

$$W_0(0) = 1 \tag{39b}$$

and

$$W_0(1) = e^{\alpha}. \tag{39c}$$

System (39), if expressed in dimensional form, is precisely that of Eq. (34) of Matovich and Pearson[7] who derive the result by presuming one-dimensional flow *a priori*.

One can obtain corrections to system (39) in powers of ϵ^2 and test for validity. Instead of doing so for fibers that are wound up at $z = 1$, we shall do so for a related fiber problem where the end at $z = 1$ is force free and the fiber flow is generated by the action of gravity acting axially along the fiber.

THE GRAVITY-DRIVEN FIBER

In this section we replace the imposed winder-speed condition (10c) by a condition of zero imposed external force. While this condition is difficult to apply in the laboratory, it is a quasisteady limit of a fiber with a free end at L (where L is increasing with time). This free-end condition takes the form

$$\frac{2}{R_0^2} \int_0^{R_0} r w_{0_z} dr = 0 \quad \text{on } z = 1. \tag{40}$$

We can find a solution of Eq. (39c) for W_0 in which the fiber experiences only viscous and gravitational forces, and satisfies at leading-order conditions (10a) and (10b) and (40). This solution is

$$w_0 = W_0(z) = \frac{\sin^2[K + z(G/6)^{1/2} \sin K]}{\sin^2 K}, \tag{41a}$$

where

$$K + (G/6)^{1/2} \sin K = \pi/2. \tag{41b}$$

To determine limits of validity on solution (41), we must calculate a first correction. This is done by paralleling the analysis of the viscous-only case. After a good deal of analysis, we find for $\overline{\mathrm{Ca}}^{-1} = \overline{\mathrm{Re}} = 0$ but $G \neq 0$ that

$$W_2'' = 2\gamma W_2' + (\gamma^2 + \delta) W_2 = \frac{1}{8}\gamma^4 + \frac{9}{2}\gamma^2\delta + 2\delta^2, \tag{42a}$$

where

$$\gamma = W_0' W_0^{-1} \tag{42b}$$

and

$$\delta = (1/6) G W_0^{-1}. \tag{42c}$$

When $G = 0$, $\gamma = \alpha$, and $\delta = 0$, Eq. (42a) reduces to form (27a). System (42) is subject here to (19a) and (19b) and the order ϵ^2 equivalent of Eq. (40) giving

$$W_2 = (1/24) W_0^{-1}(G + 6W_0'^2 W_0^{-1}) \quad \text{on } z = 0 \tag{42d}$$

and

$$W_2' = (1/8) W_0' W_0^{-2} (4W_0' W_0^{-1} - G) \quad \text{on } z = 1. \tag{42e}$$

We have solved system (42) numerically for various values of G and, using the free-end equivalent of relation (26), obtained R_2. The plots shown in Figure 2 show that $|R_2|$ is largest close to $z = 0$. It can be seen that when G becomes large, say greater than 100, the value of R_2 is comparable to R_0 and ϵ^2 must be very small for the correction to be small. If we use criterion (29) for neglect of the corrections, we find that we must have

$$(1/4)\epsilon^2 W_0^{-2} \left[\frac{1}{3} G + 2W_0'^2 W_0^{-1}\right] \ll 1. \tag{43a}$$

If (43a) is evaluated at $z = 0$, the inequality is guaranteed if

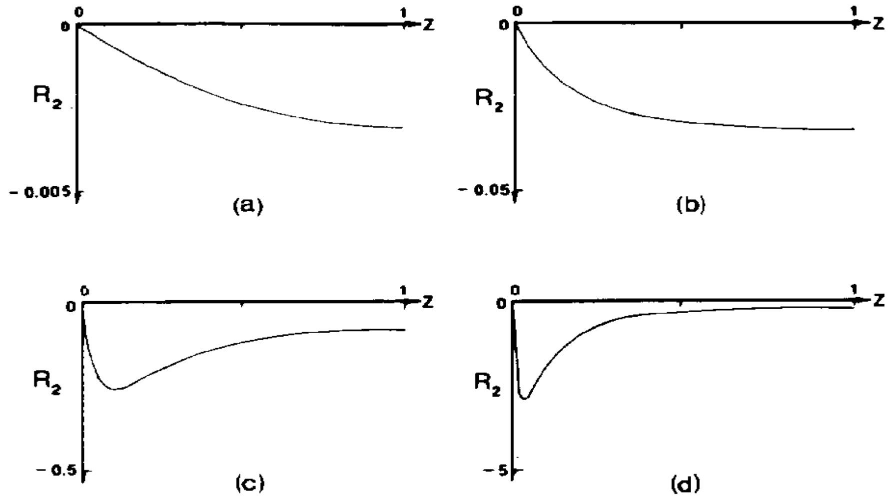

Fig. 2. Numerical solutions for $R_2(z)$ of system (42) for (a) $G = 1$, (b) $G = 10$, (c) $G = 100$, (d) $G = 1000$.

$$(1/12)\epsilon^2 G \ll 1, \tag{43b}$$

which translates through definition (33) to

$$\mathrm{Re}/\mathrm{Fr} \ll 12. \tag{43c}$$

Unlike the condition (31) for validity of the one-dimensional solutions for fibers with a winder and $G = 0$, condition (43) can be *very restrictive.* The value of Re/Fr must be unity or smaller or the radial variation in the axial velocity becomes appreciable and the standard one-dimensional model breaks down.

DISCUSSION AND CONCLUSIONS

We have used a procedure to derive systematically the one-dimensional approximate systems plus their corrections from the steady, free-surface, axisymmetric equations of a liquid jet. These corrections allow parameter estimates for the validity of the one-dimensional flows. Only the "outer" region sufficiently far from the fiber ends has been treated carefully, the "inner" boundary-layer behavior near the orifice has been studied numerically by Fisher, Denn, and Tanner.[2] They find that the lowest-order one-dimensional solution is valid to within one diameter of the orifice.

We have discussed two types of fiber systems, viz., ones having applied tensions at their ends and ones having free ends but driven by axial gravity. A fairly strict limitation on the magnitude of gravity term was found to be consistent with one-dimensional flow assumptions.

We have examined here only the simplest of fiber systems. That is, steady, Newtonian liquid jets with uniform properties that enter into passive gaseous environments. Thus, our conclusions on the validity of one dimensionality are limited by these features. However, the *procedure* presented here is a basis for treating much more elaborate fiber systems, especially nonisothermal jets and polymer jets having complex rheological behavior.

The authors wish to acknowledge the support of Owens-Corning Fiberglas Corporation. They also thank Dr. N. E. Greene for critically reading an earlier version of this article.

References

1. R. E. Nickell, R. I. Tanner, and B. Caswell, *J. Fluid Mech.*, **65,** 189 (1974).
2. R. J. Fisher, M. M. Denn, and R. I. Tanner, *Ind. Eng. Chem. Fundam.*, **19,** 195 (1980).
3. J. Gavis and M. Modan, *Phys. Fluids*, **10,** 487 (1967).
4. B. A. Whipple and C. T. Hill, *AIChE J.*, **24,** 664 (1978).
5. F. T. Trouton, *Proc. R. Soc. London Ser. A.*, **77,** 426 (1906).
6. L. R. Glicksman, *J. Basic Eng.*, **90,** 343 (1968).
7. M. A. Matovich and J. R. A. Pearson, *Ind. Eng. Chem. Fundam.*, **8,** 512 (1969).
8. N. S. Clarke, *Quart. J. Mech. Appl. Math.*, **22,** 247 (1969).
9. M. M. Denn, *Ann. Rev. Fluid Mech.*, **12,** 365 (1980).
10. F. T. Geyling, *Bell Sys. Tech. J.*, **55,** 1011 (1976).
11. S. Kase, *J. App. Polym. Sci.*, **18,** 3267 (1974).
12. A. Kaye and D. G. Vale, *Rheol. Acta*, **8,** 1 (1969).
13. R. J. Fisher and M. M. Denn, *Chem. Eng. Sci.*, **30,** 1129 (1975).

Received September 9, 1981
Accepted as revised February 4, 1982