

BRIEF COMMUNICATIONS

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Taylor vortices between elliptical cylinders

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Critical parameters (Reynolds number and wave number) signaling the onset of Taylor vortices are calculated for the flow between “elliptical” cylinders. The spinning inner cylinder is circular; the stationary outer cylinder is composed of two circular arcs and is similar to an ellipse. It is shown that increasing ellipticity destabilizes the flow and increasing eccentricity stabilizes the flow. The spectral element method is used to calculate the base flow and to solve the linear stability problem.

In this Brief Communication, we describe the linear stability of flow between a circular inner cylinder spinning with an angular speed ω and a stationary outer “elliptical” cylinder. Both cylinders are considered infinitely long. As seen in Fig. 1, the outer cylinder is not actually an ellipse, but an oblong shape constructed from two circular arcs of equal size. The use of “elliptical” to describe such a geometry is common in lubrication literature. For small clearances between the cylinders, this geometry resembles that of an elliptical journal bearing (actual bearings have finite length and are often fitted with lubricant inlet and outlet flow ports). Pinkus^{1,2} and Kumar *et al.*³ have studied torque and load characteristics based on solution of the lubrication equation.⁴ We discuss here the flow instability characterized by the appearance of Taylor vortices for the constrained location of the cylinder centers. Another type of bearing instability known as “shaft whirl” or “oil whip” arises from interaction between fluid forces and shaft translation and is discussed in Ref. 5. The increased shaft stability of an elliptical journal bearing makes it attractive for high-speed applications, such as in turbines or rotary compressors.⁶

The Taylor–Couette stability problem for flow between concentric circular cylinders has been extensively studied (see the review article by DiPrima and Swinney⁷). This flow lends itself to analysis because the base flow is one dimensional and the exchange-of-stabilities principle can be used. When the cylinders are eccentrically placed and/or of arbitrary shape, the base flow is Reynolds number dependent and exchange-of-stabilities cannot be assumed, substantially complicating the analysis. DiPrima and Stuart⁸ performed a perturbation analysis for eccentric circular cylinders assuming small clearance and eccentricity.

They showed that increasing eccentricity stabilizes the flow. Subsequent papers by DiPrima and Stuart⁹ and Eagles *et al.*¹⁰ investigated the nonlinear characteristics of Taylor vortices between eccentric circular cylinders. Oikawa *et al.*¹¹ numerically investigated the linear stability of the flow between eccentric circular cylinders without the restrictions of small eccentricity or clearance. Oikawa’s results show good agreement with the DiPrima and Stuart analysis for small eccentricity and clearance. They also agree well with experiments by Vohr¹² and Karasudani.¹³

There are no analytical or numerical studies of the stability of flows between noncircular cylinders. Lewis¹⁴ solved for the base flow between a rotating circular inner cylinder and fixed square outer cylinder using finite differences. Snyder,¹⁵ Mullin,¹⁶ and Terada and Hattori¹⁷ experimentally studied flow stability for a variety of inner/outer noncircular cylinder arrangements and characterized the bifurcation to Taylor vortices for the various geometries.

The two nondimensional parameters describing the concentric elliptical geometry are the ellipticity ratio, $e^* = e/d$, and the minimum gap radius ratio, $\eta = R_i/(R_i + d)$, where e is the distance between the inner cylinder center and the center of the circle describing either lobe of the outer cylinder, d is the minimum gap width, and R_i is the inner cylinder radius (see Fig. 1).

We use the spectral element method^{5,18–20} to solve the base flow and linear stability problems. For the two-dimensional base flow, we solve the incompressible Navier–Stokes equations in a Cartesian coordinate system:

$$\text{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} (\mathbf{u} \cdot \nabla \mathbf{u} + \nabla \cdot \mathbf{u} \mathbf{u}) \right) = \nabla^2 \mathbf{u} - \text{Re} \nabla p, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

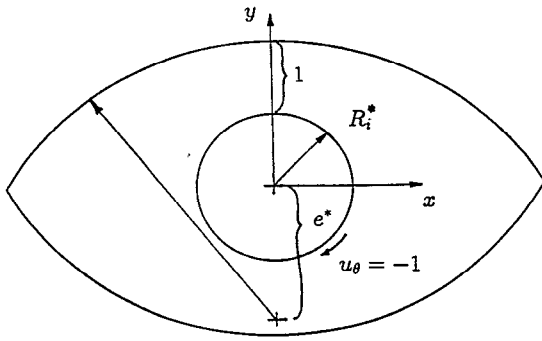


FIG. 1. Elliptical geometry. All parameters shown are dimensionless.

The Reynolds number is $Re = \omega R d / \nu$, where ν is the kinematic viscosity. The skew-symmetric form for the nonlinear terms is used to minimize aliasing errors.²¹ The unknowns within an element are approximated in a series of orthogonal polynomials:

$$\mathbf{u} = \sum_{m=0}^N \sum_{n=0}^N \mathbf{u}_{mn} h_m(r) h_n(s) \quad (3)$$

and

$$p = \sum_{m=0}^{N-2} \sum_{n=0}^{N-2} p_{mn} L_m(r) L_n(s), \quad (4)$$

where $h_m(r)$ is the Lagrangian interpolant of order N constructed from Legendre polynomials, $L_m(r)$ is the Legendre polynomial of order m , \mathbf{u}_{mn} is the velocity at the m th grid point, p_{mn} is the m th spectral coefficient for pressure, and r and s are the computational coordinates related to the Cartesian coordinates by

$$\mathbf{x} = \sum_{m=0}^N \sum_{n=0}^N \mathbf{x}_{mn} h_m(r) h_n(s). \quad (5)$$

The pressure approximation is two orders less than the velocity approximation to avoid spurious pressure modes.¹⁹ The formulation consists of multiplying (1) and (2) by appropriate test functions, integrating by parts, inserting the approximations (3) and (4), and performing Gauss-Lobatto quadrature.^{18,22} We use an implicit time-marching scheme that allows large time steps to rapidly find steady-state solutions.⁵

For the stability analysis, we assume small perturbations of the form

$$\begin{pmatrix} u \\ v \\ w \\ p \end{pmatrix} (x, y, z, t) = \begin{pmatrix} U \\ V \\ 0 \\ P \end{pmatrix} (x, y) + e^{\sigma t} \begin{pmatrix} \hat{u} \cos kz \\ \hat{v} \cos kz \\ \hat{w} \sin kz \\ \hat{p} \cos kz \end{pmatrix}, \quad (6)$$

where k is the real wave number along the z axis and σ is (in general) a complex growth rate. Substituting (6) into the nonconservative form of the three-dimensional Navier-Stokes equations and linearizing yields

$$\begin{aligned} \text{Re}(\sigma \hat{\mathbf{u}} + \mathbf{U} \cdot \nabla \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \mathbf{U}) &= -\text{Re} \nabla^2 \hat{\mathbf{u}} + \nabla^2 \hat{\mathbf{u}} - k^2 \hat{\mathbf{u}}, \\ \text{Re}(\sigma \hat{w} + \mathbf{U} \cdot \nabla \hat{w}) &= \text{Re} k \hat{p} + \nabla^2 \hat{w} - k^2 \hat{w}, \\ \nabla \cdot \hat{\mathbf{u}} + k \hat{w} &= 0. \end{aligned} \quad (7)$$

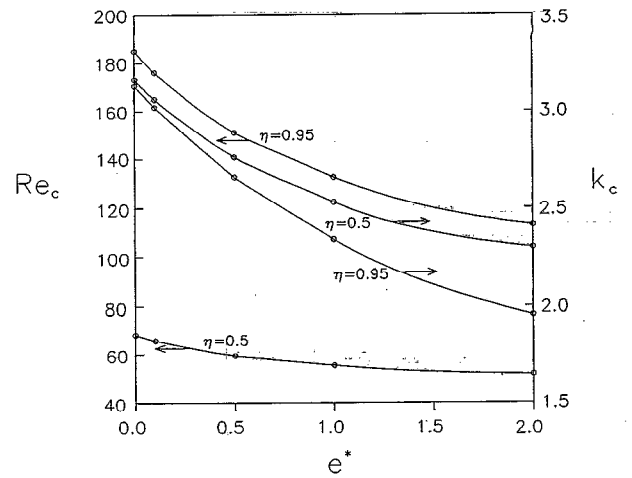


FIG. 2. Critical Reynolds number and wave number versus ellipticity for $\eta = 0.5$ and 0.95 .

where

$$\hat{\mathbf{u}} = \hat{u} \hat{\mathbf{i}} + \hat{v} \hat{\mathbf{j}}, \quad \mathbf{U} = U \hat{\mathbf{i}} + V \hat{\mathbf{j}}. \quad (8)$$

The discretization parallels that performed for the base flow and leads to the following generalized eigenvalue problem:⁵

$$\mathbf{G} \mathbf{x} = \sigma \mathbf{B} \mathbf{x}, \quad (9)$$

where $\sigma = f(Re, k, e^*, \eta)$. The vector \mathbf{x} is comprised of the unknown velocity and pressure coefficients, and \mathbf{B} is a square diagonal matrix whose entries corresponding to the continuity equation and the homogeneous boundary conditions are zero. The zeros in the matrix \mathbf{B} result in “infinite” eigenvalues that preclude the use of a standard power routine to isolate the most dangerous eigenvalue (i.e., the eigenvalue with the largest real part).^{23,24} We therefore choose to solve the eigenvalue problem directly with the EISPACK routine RGG. Since the base flow is Reynolds number dependent, the critical parameters Re_c and k_c must be determined in an iterative fashion. Note that Re_c and k_c correspond to the real part of σ being equal to zero.

We tested our code by confirming the critical parameters found by Oikawa *et al.*¹¹ for eccentric circular cylinders, including an instance of a Hopf bifurcation.⁵ To avoid degradation of spectral accuracy due to the sharp corners in the elliptical cylinder problem, we divide the flow do-

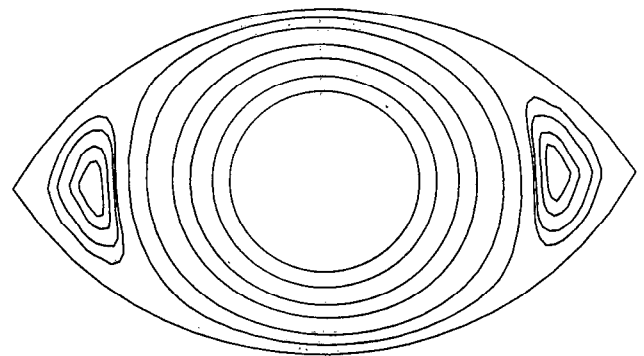


FIG. 3. Streamlines for base flow. $Re = 51.75$, $\eta = 0.5$, $e^* = 2$.

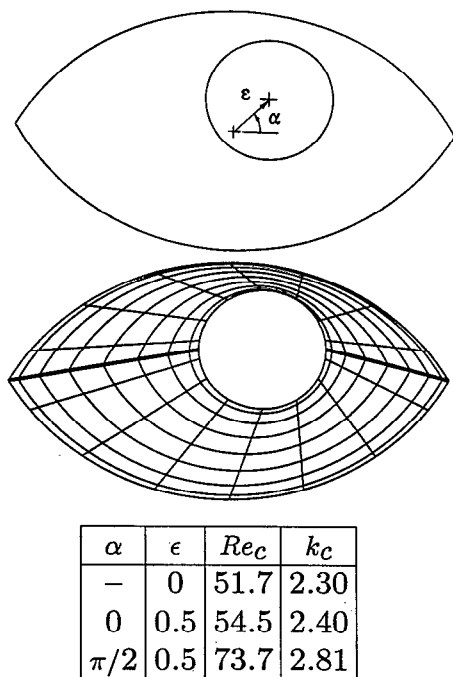


FIG. 4. Geometry, typical grid ($N=8$), and critical parameters for eccentric elliptical cylinders. $\eta=0.5$, $e^*=2$.

main into two elements such that the element corners coincide with the outer cylinder corners.

Figure 2 shows how the critical Reynolds number and wave number change with ellipticity e^* . We show results for two minimum gap radius ratios: $\eta=0.5$ and $\eta=0.95$. For all the $\eta=0.5$ runs, we used a square truncation with $N=10$ in each of the two elements. For $\eta=0.95$, we used $N=10$ for $e^*=0.1$ and 0.5 , and $N=14$ for $e^*=1$ and 2 . We ensured convergence of the eigenvalues by confirming the eigenvalues with higher truncations.

Figure 3 shows streamlines for the base flow with $e^*=2$ and $\eta=0.5$ at a slightly supercritical state where $Re=51.75$. The truncation is $N=10$. Analysis showed that the slight irregularities in the illustrated corner vortices are due not to poor resolution of the base flow velocity solution, but to limitations in our contour-plotting routine.

As the inner cylinder is moved from the concentric position, it might be expected that the critical Reynolds number and wave number would increase. This would be consistent with the results for eccentric circular cylinders from Oikawa *et al.*¹¹ With eccentric elliptical cylinders, two additional parameters are introduced: the eccentricity ratio ϵ and the angle α , as seen in Fig. 4. The eccentricity ratio is defined as $\epsilon=\hat{e}/d$, where \hat{e} is the distance between the inner and outer cylinder centers. Note that, for any angles other than $\alpha=\pm\pi/2$, ϵ can exceed unity. In Fig. 4, we tabulate three cases: $\epsilon=0$; $\epsilon=0.5$, $\alpha=\pi/2$; and $\epsilon=0.5$, $\alpha=0$. As expected, the critical parameters for both eccentric cases are higher than those for the concentric case. For a given eccentricity, the flow is more stable for angles that result in smaller clearances.

In conclusion, we find that increasing ellipticity destabilizes the flow. It is well known that, for the concentric cylinder problem, wider gaps are associated with lower

critical Reynolds numbers. Since the appearance of Taylor vortices may be dictated by an average gap in the elliptical cylinder case, the decreasing critical Reynolds number with increasing ellipticity makes sense. This is, however, in contrast to Snyder's¹⁵ experimental observation that, for a spinning inner circular cylinder and stationary outer square cylinder arrangement, the appearance of vortices was dictated by the *minimum* gap.

Increasing ellipticity widens the Taylor vortices in the z direction. Snyder noted that the wavelength of the vortices in his geometry lay somewhere between twice the minimum and maximum gap widths. We find the same phenomena here, as is evident by the increasing ellipticity/decreasing wave-number relationship shown in Fig. 2.

As with eccentric circular cylinders, increasing eccentricity stabilizes the flow and results in smaller vortices. For given eccentricity, ellipticity, and η , the flow is more stable for angles that result in smaller clearances between the two cylinders.

We found no Hopf bifurcation for the elliptical geometry results presented here. However, since Hopf bifurcations do occur for eccentric cylinders with small gap and high eccentricity,¹¹ we suspect that at large enough values of η and e^* a Hopf bifurcation will also occur for elliptical geometries. Computer resource limitations prevented us from trying the necessary higher truncations. Perhaps a mapping that stretches the radial coordinate in the narrowest parts of the flow field would allow solutions with reasonable truncations.

ACKNOWLEDGMENT

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